A GEOMETRIC APPROACH TO MULTI-MODAL AND MULTI-AGENT SYSTEMS

FROM DISTURBANCE DECOUPLING TO CONSENSUS

ANNEROOS R.F. EVERTS

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1 INTRODUCTION

Dynamical systems with nonsmooth components arise in various settings. For example, nonsmoothness may occur due to dry friction, impacts or backlash in mechanical systems, or due to diode elements in electrical circuits. The governing differential equations of nonsmooth dynamical systems have right-hand sides which are not differentiable or even discontinuous. Consequently, the classical control theory for smooth systems cannot be applied; a different and more careful approach is needed. This approach includes extending differential equations to differential inclusions and adopting different solution concepts.

In this thesis, we focus our attention on two types of nonsmooth dynamical systems. First, we study linear multi-modal systems. These systems consist of a collection of linear systems, each of which is active on a polyhedral region of the state space. Although these systems exhibit linear behavior locally, they are nonsmooth, since their vector fields lack differentiability. Second, we study multi-agent systems with a nonlinear communication protocol. The nonlinear functions in these systems do not have the property that they are locally linear, but they do satisfy a property that we call sign-preservation.

In the rest of this chapter, we have a closer look at both types of nonsmooth dynamical systems and we introduce the problems that are studied in this thesis for these systems.

1.1 LINEAR MULTI-MODAL SYSTEMS

Linear multi-modal systems form a class of hybrid systems; they are a combination of continuous-time linear systems, the *modes*, together with the discrete dynamics of switching between these modes. In this thesis, we study two general classes of linear multi-modal systems.

First, we consider continuous piecewise affine systems. The state space of a piecewise affine system is divided into solid polyhedral regions, on each of which a different linear or affine system is active. We call the regions with their corresponding linear systems the modes. Switching between different modes of a piecewise affine system is state-dependent: if the state lies in a certain region, then the corresponding linear or affine system is active at that moment. The resulting vector field is not differentiable, but we do assume continuity of the right-hand side of the governing differential equations. Piecewise affine systems can be used to approximate nonlinear systems, but they can also appear naturally, for example in systems that deal with friction.

Second, we generalize piecewise linear systems to linear multi-modal systems for which the polyhedral regions may overlap and do not have to cover the full state space. As a consequence, we have to replace the differential equations by differential inclusions. Examples of such linear multi-modal systems are switched linear systems, conewise linear systems, and linear complementarity systems.

In this thesis, we study two geometric control theory problems for linear multi-modal systems, namely the disturbance decoupling problem and the fault detection and isolation problem.

1.2 THE DISTURBANCE DECOUPLING PROBLEM

Annihilating or reducing the effects of disturbances is of major importance in many real-life control problems. Designing feedback laws that decouple the disturbances from a certain to-be-controlled output constitutes the well-known disturbance decoupling problem. An input/state/output system is called *disturbance decoupled* if for each fixed initial condition and zero input, the output corresponding to one disturbance is exactly the same as the output corresponding to another disturbance. The disturbance decoupling problem amounts to finding a feedback law that renders the system disturbance decoupled by eliminating the effect of disturbances on the output. The investigation of this problem for linear and (smooth) nonlinear systems has been the starting point for the development of geometric control theory [Basile and Marro, 1969a,b; Wonham and Morse, 1970]. For both linear and (smooth) nonlinear systems, geometric control theory has been proven to be very efficient in solving various control problems, including the disturbance decoupling problem (see e.g. [Wonham, 1985; Nijmeijer and van der Schaft, 1990; Basile and Marro, 1992; Isidori, 1995; Trentelman et al., 2001]). In this thesis, we use tools from geometric control theory to study the disturbance decoupling problem for both piecewise affine systems and the more general linear multi-modal systems.

So far, in the context of hybrid dynamical systems, the results on the disturbance decoupling problem are limited to jumping hybrid systems [Conte et al., 2015] and switched linear systems [Conte et al., 2014; Otsuka, 2010, 2011, 2015; Yurtseven et al., 2012; Zattoni and Marro, 2013; Zattoni et al., 2016]. The major difference between piecewise affine systems and switched linear systems is the nature of the switching behavior. For piecewise affine systems the switching behavior is state-dependent whereas it is state-independent for switched linear systems.

For the case of state-independent switching, the solution of the disturbance decoupling problem can be obtained by following mainly the footsteps of the (non-switching) linear case. An interesting consequence of the state-independent switching is that the set of reachable states under the influence of disturbances is a subspace. This allows one to generalize the so-called controlled invariant subspaces of linear systems to switched linear systems. Such a generalization leads to elegant necessary and sufficient conditions [Otsuka, 2010; Yurtseven et al., 2012] for a switched linear system to be disturbance decoupled. In the same papers, disturbance decoupling problems by different feedback schemes have also been solved based on these necessary and sufficient conditions.

However, a similar approach breaks down in the case of state-dependent switching as the set of reachable states under the influence of disturbances is not anymore a subspace, not even a convex set in general. As such, neither the results nor the approach adopted for the state-independent case can be applied to the state-dependent switching case.

In this thesis, we develop a new approach that takes into account the state-dependent switching behavior of piecewise affine systems. In Chapter 2, based on the conference paper [Everts and Camlibel, 2014b], we start by studying a simple class of piecewise affine systems, namely bimodal linear systems. Our approach provides easily verifiable geometric necessary and sufficient conditions for these systems to be disturbance decoupled. Based on these conditions, we study the disturbance decoupling problem for both state feedback controllers and dynamic feedback controllers. For both feedback schemes, we consider mode-independent and mode-dependent controllers, and provide necessary and sufficient conditions for the solvability of the disturbance decoupling problem. These conditions amount to checking certain subspace inclusions very much analogous to linear systems and linear state-independent switching systems.

In Chapter 3, we study the disturbance decoupling problem for general continuous piecewise affine systems, based on the conference paper [Everts and Camlibel, 2014a]. We provide a set of necessary conditions and a set of sufficient conditions under which such a system is disturbance decoupled. Although these conditions do not coincide in general, we point out some special cases in which they do coincide. Furthermore, we present conditions for the existence of mode-independent static feedback controllers that render the closed-loop system disturbance decoupled. All the conditions we present are geometric in nature and easily verifiable.

Next, we consider a particular linear complementarity system and study when such a system is disturbance decoupled in Chapter 4, based on the book chapter [Everts and Camlibel, 2015]. Linear complementarity systems are nonsmooth dynamical systems that are obtained by taking a standard linear input/output system and imposing certain complementarity relations on a number of input/output pairs at each time instant. A wealth of examples, from various areas of engineering as well as operations research, of linear complementarity systems can be found in [Camlibel et al., 2004; Schumacher, 2004; van der Schaft and Schumacher, 2000; Heemels and Brogliato, 2003]. For the work on the analysis of linear complementarity systems, we refer to [Camlibel et al., 2003; Heemels et al., 2002; Camlibel et al., 2002; van der Schaft and Schumacher, 1996; Camlibel, 2007; van der Schaft and Schumacher, 1998; Heemels et al., 2000]. Particular linear complementarity systems can be written as linear multi-modal systems, namely those of index zero [Camlibel, 2001, Chapter 2]. Different from the piecewise affine systems treated before, the polyhedral regions on which the modes are active, can now be non-solid, and together they do not cover the full state space. It turns out that the resulting linear subsystems share a certain geometric structure, which we exploit to find necessary and sufficient conditions for disturbance decoupledness that are crisp and easily checkable.

Finally, in Chapter 5, we study the disturbance decoupling problem for general linear multi-modal systems. We recover almost all results from Chapters 2, 3 and 4, as well as known results for a particular class of switched linear systems. Moreover, this general approach gives us the possibility to treat the disturbance decoupling problem for another class of linear complementarity systems, namely linear passive-like complementarity systems [Camlibel, 2001; Camlibel et al., 2002; Camlibel and Schumacher, 2016]. We find novel necessary and sufficient conditions for this kind of linear complementarity systems to be disturbance decoupled. Chapter 5 is based on the journal paper [Everts and Camlibel, 2016].

1.3 THE FAULT DETECTION AND ISOLATION PROBLEM

The second geometric control problem that we consider in this thesis is the fault detection and isolation problem, which we first study for bimodal piecewise linear systems. Given a system that is prone to faults, the fault detection and isolation problem amounts to finding an observer that detects when a faults occurs. Moreover, if a fault occurs then the observer should identify what kind of fault it is. Examples of faults are the complete failure of an actuator, a biased actuator, or changes in the system dynamics.

Fault detection and isolation (FDI) is an active area of research in control theory, due to the essential requirement of high reliability for many applications of control systems. Various types of FDI techniques have been proposed for linear systems and for some classes of nonlinear ones. There is a large number of contributions in this area, and consequently we direct the interested reader to the comprehensive survey papers [Frank, 1990; Hwang et al., 2010; Isermann, 2006; Isermann and Bailé, 1997]. On the other hand, research on FDI for hybrid and switched systems, and in particular for piecewise linear systems, has been less intensive and fruitful (see [Balluchi et al., 2002; Cocquempot et al., 2004; Narasimhan et al., 2000; Wang et al., 2009]).

In Chapter 6 of this thesis, based on the paper [Everts et al., 2016], we use the classical geometric control theory framework (see [Basile and Marro, 1992; Wonham, 1985]) to investigate the problem of fault detection and isolation for bimodal linear systems. Our approach is inspired by the ideas pioneered in [Massoumnia, 1986a], where several formulations of the fault detection and isolation problem were stated and solved in geometric terms for linear systems. We give a sufficient condition for solving the fault detection and isolation problem for bimodal linear systems.

As a by-product, and before we continue with the second class of nonsmooth dynamical systems, we consider the FDI problem for a class of linear dynamical systems defined over an undirected graph. Two disjoint sets of agents are identified in the network: the faultable agents, which are prone to failure, and the observer agents, whose output is measurable. Fault detection is performed by an unknown input observer, and stated in the geometric language of [Massoumnia, 1986a], i.e. output separability of fault subspaces. In Chapter 7, which is based on the paper [Rapisarda et al., 2015], we present a characterization of the smallest conditioned invariant subspaces that are generated by the faults. This characterization is exploited in order to give graph-theoretical conditions guaranteeing output separability in terms of distances between faultable agents and observer ones. In addition, we study the case where two faultable vertices share exactly the same neighbors in order to present a condition under which fault detectability fails.

In this thesis, we make extensive use of geometric control theory for both the disturbance decoupling problem and the FDI problem. Although the resulting conditions often are somewhat similar to those of the linear case, the methods to obtain these conditions are fundamentally different, since we have to take state-dependent switching into account.

As a last subject in this thesis, we study a truly nonlinear system, that does not exhibit linear behavior locally.

1.4 NONLINEAR CONSENSUS PROTOCOLS FOR DIGRAPHS

The second class of nonsmooth dynamical systems that we consider in this thesis arises in the context of nonlinear consensus protocols. We consider a network of agents that communicate according to a fixed communication topology, represented by a directed graph containing a directed spanning tree. For the well-known linear consensus protocol these graph-theoretical are known to be a sufficient and necessary condition for reaching state consensus. In Chapter 8 of this thesis, we generalize this result and study the consensus problem for a general nonlinear consensus protocol. A nonlinear consensus protocol may arise due to the nature of the controller [Jafarian and De Persis, 2015; Saber and Murray, 2003], or may describe the physical coupling existing in the network [Bürger et al., 2014; Monshizadeh and De Persis, 2015]. The nonlinear functions in our model are assumed to be sign-preserving and are allowed to have possible discontinuities. Examples of such functions are the saturation function and the sign function. To deal with the possible discontinuities, we will need to employ Filippov solutions and replace the differential equations by differential inclusions [Filippov, 1988; Smirnov, 2002; Aubin and Cellina, 1984]. In this framework, we first study the case that these nonlinearities happen at the level of the nodes. Next, we consider the case that we have nonlinearities at the edges. Finally, we combine these results. Chapter 8 is based on the paper [Wei et al., 2016].

The last section of this chapter is devoted to discussing some notation and notions in geometric control theory, which we use throughout this thesis.

1.5 PRELIMINARIES AND NOTATION

First, we fix some notation and definitions that we use throughout this thesis. Let $\mathbb{R}$ denote the set of real numbers. For a vector v we denote its transpose by v^T and its dimension by n_v. For two vectors v and w, we let $\operatorname{col}(v, w)$ denote the column vector that is obtained by stacking v and w. We denote the set $\{1, 2, \ldots, m\}$ by $\mathcal{I}_m$. For a subset α of $\mathcal{I}_m$, α^c denotes the subset $\mathcal{I}_m \setminus \alpha$. A *cone* is a subset of a vector space that is closed under multiplication by positive scalars.

Let S be a set in $\mathbb{R}^n$. The Minkowski sum of two subsets S_1 and S_2 of S is given by

$$S_1 + S_2 = \{x + y \mid x \in S_1, y \in S_2\}.$$

The affine hull of S is the smallest affine set containing S in $\mathbb{R}^n$ and is denoted by $\operatorname{aff}(S)$. The *relative interior* of S is defined as

$$\operatorname{rint}(S) := \{x \in S : \exists \epsilon > 0, N_\epsilon(x) \cap \operatorname{aff}(S) \subseteq S\},$$

where $N_\epsilon(x)$ is an ϵ-neighborhood of x. We call the set S *solid* if the affine hull of $\operatorname{rint}(S)$ is n-dimensional. With $S^\perp$ we denote the orthogonal complement of S with respect to the inner product $v^\mathsf{T} w$ for $v, w \in \mathbb{R}^n$.

1.5.1 *Geometric control theory*

Geometric control theory, illustrated in depth in e.g. [Basile and Marro, 1992] and [Trentelman et al., 2001], plays an important role in this thesis. The rest of this section quickly summarizes some definitions and results from geometric control theory that are relevant to this thesis.

Consider the linear, time-invariant system $\Sigma = \Sigma(A, B, C, D)$ given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1a}$$
$$y(t) = Cx(t) + Du(t), \tag{1.1b}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the input, $y \in \mathbb{R}^{n_y}$ is the output, and A, B, C and D are matrices of appropriate dimensions.

A subspace $\mathcal{V} \subseteq \mathbb{R}^{n_x}$ is called *A-invariant* if $\mathcal{V}$ satisfies $A\mathcal{V} \subseteq \mathcal{V}$. The *controllable subspace* is the smallest A-invariant subspace containing $\operatorname{im} B$. It is denoted by $\langle A \mid \operatorname{im} B\rangle$ and satisfies

$$\langle A \mid \operatorname{im} B\rangle = \operatorname{im} B + \operatorname{im} AB + \cdots + \operatorname{im} A^{n_x-1}B. \tag{1.2}$$

Note that for any matrix $K \in \mathbb{R}^{n_u \times n_x}$ we have

$$\langle A + BK \mid \operatorname{im} B\rangle = \langle A \mid \operatorname{im} B\rangle. \tag{1.3}$$

A subspace $\mathcal{V} \subseteq \mathbb{R}^{n_x}$ is called *controlled invariant* with respect to A and B, or *(A, B)-invariant* in short, if there exists a matrix F such that $\mathcal{V}$ is $(A + BF)$-invariant. Such a matrix F is called a *friend* of $\mathcal{V}$. Equivalently, a subspace is controlled invariant if

$$A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B. \tag{1.4}$$

From (1.4) it follows that the sum of two controlled invariant subspaces is again controlled invariant.

A subspace $\mathcal{V} \subseteq \mathbb{R}^{n_x}$ is called an *output nulling controlled invariant subspace* of Σ if

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq \big(\mathcal{V} \times \{0\}\big) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}.$$

The *weakly unobservable subspace* of Σ is the largest (with respect to the subspace inclusion) output nulling controlled invariant subspace and will be denoted by $\mathcal{V}^*(\Sigma)$. In the case that $D = 0$, we sometimes write $\mathcal{V}^*(C, A, B)$ to denote $\mathcal{V}^*(\Sigma)$, which is then the largest (A, B)-invariant subspace contained in $\ker C$.

A subspace $\mathcal{T} \subseteq \mathbb{R}^{n_x}$ is called *conditioned invariant* with respect to C and A, or *(C, A)-invariant* in short, if

$$A(\mathcal{T} \cap \ker C) \subseteq \mathcal{T}.$$

This condition is equivalent to the existence of a matrix $K \in \mathbb{R}^{n_x \times n_y}$ such that $\mathcal{T}$ is $(A + KC)$-invariant, i.e.

$$(A + KC)\mathcal{T} \subseteq \mathcal{T}.$$

We call such a matrix K a *friend* of $\mathcal{T}$. The intersection of two conditioned invariant subspaces is again conditioned invariant.

We call a subspace $\mathcal{T} \subseteq \mathbb{R}^{n_x}$ *input containing conditioned invariant* if

$$\begin{bmatrix} A & B \end{bmatrix} \left((\mathcal{T} \times \mathbb{R}^{n_u}) \cap \ker \begin{bmatrix} C & D \end{bmatrix} \right) \subseteq \mathcal{T}.$$

It is well-known that a subspace $\mathcal{T}$ is an input containing conditioned invariant subspace if and only if there exists a matrix $L \in \mathbb{R}^{n_x \times n_y}$ such that

$$(A + LC)\mathcal{T} \subseteq \mathcal{T} \quad \text{and} \quad \operatorname{im}(B + LD) \subseteq \mathcal{T}. \tag{1.5}$$

The *strongly reachable subspace* of Σ is defined to be the smallest (with respect to the subspace inclusion) input containing conditioned invariant subspace and will be denoted by $\mathcal{T}^*(\Sigma)$.

Let K and L be $m \times n$ and $n \times p$ matrices, respectively. Denote the system $\Sigma(A + BK + LC + LDK, B + LD, C + DK, D)$ by $\Sigma_{K,L}$. Then we have the following equality:

$$\mathcal{T}^*(\Sigma_{K,L}) = \mathcal{T}^*(\Sigma). \tag{1.6}$$

It follows from (1.5) with the choice of $L = 0$ that the controllable subspace is an input containing conditioned invariant subspace. Hence, we have

$$\mathcal{T}^*(\Sigma) \subseteq \langle A \mid \operatorname{im} B \rangle. \tag{1.7}$$

In the case that $D = 0$, we sometimes write $\mathcal{T}^*(B, C, A)$ to denote $\mathcal{T}^*(\Sigma)$, which is then the smallest (C, A)-invariant subspace containing $\operatorname{im} B$. It can be shown that for a friend K of $\mathcal{T}^*(B, C, A)$, we have $\mathcal{T}^*(B, C, A) = \langle A + KC \mid \operatorname{im} B \rangle$. The subspace $\mathcal{T}^*(B, C, A)$ can be computed by the following subspace algorithm (see e.g. Algorithm 4.1.1 p. 203 of [Basile and Marro, 1992]):

$$\mathcal{T}^0 := \operatorname{im} B \tag{1.8a}$$

$$\mathcal{T}^k := \operatorname{im} B + A \left(\mathcal{T}^{k-1} \cap \ker C \right) \tag{1.8b}$$

for $k \geqslant 1$. As these subspaces are nested, that is

$$\mathcal{T}^k \subseteq \mathcal{T}^{k+1},$$

it follows that there exists an integer ℓ such that $0 \leqslant \ell \leqslant n$ and

$$\mathcal{T}^\ell = \mathcal{T}^{\ell+1}.$$

It is well-known that

$$\mathcal{T}^*(B, C, A) = \mathcal{T}^\ell.$$

A pair of subspaces $(\mathcal{T}, \mathcal{V})$ is called a (C, A, B)*-pair* if $\mathcal{T}$ is (C, A)-invariant, $\mathcal{V}$ is (A, B)-invariant and $\mathcal{T} \subseteq \mathcal{V}$. If $(\mathcal{T}, \mathcal{V})$ is a (C, A, B)-pair, then there is a linear map $N : \mathbb{R}^{n_y} \to \mathbb{R}^{n_u}$ such that $(A + BNC)\mathcal{T} \subseteq \mathcal{V}$ (see e.g. [Trentelman et al., 2001, Lemma 6.3]).

It is well-known that the transfer matrix $D + C(sI - A)^{-1}B$ is right-invertible as a rational matrix if and only if

$$\mathcal{V}^*(\Sigma) + \mathcal{T}^*(\Sigma) = \mathbb{R}^{n_x} \text{ and } \begin{bmatrix} C & D \end{bmatrix} \text{ is of full row rank.}$$

Straightforward linear algebra arguments (see e.g. [Kaba, 2001, Ch. 2, Thm. 4] and [Trentelman et al., 2001, Thm. 8.27]) show that these conditions are equivalent to

$$\operatorname{im} D + C\mathcal{T}^*(\Sigma) = \mathbb{R}^{n_y}. \tag{1.9}$$

2

DISTURBANCE DECOUPLING FOR CONTINUOUS PIECEWISE LINEAR BIMODAL SYSTEMS

ABSTRACT: *In this chapter we tackle the disturbance decoupling problem for continuous bimodal piecewise linear systems. After establishing necessary and sufficient geometric conditions for such a system to be disturbance decoupled, we study state feedback and dynamic feedback controllers, both mode-dependent and mode-independent. For these feedback schemes, we provide necessary and sufficient conditions for the solvability of the disturbance decoupling problem. Also, we provide subspace algorithms in order to verify the presented conditions. This chapter is based on the conference paper [Everts and Camlibel, 2014b].*

2.1 INTRODUCTION

One of the main problems that will be addressed in this thesis is the disturbance decoupling problem for piecewise affine systems and other linear multi-modal systems. As introduced in Chapter 1, the disturbance decoupling problem amounts to eliminating, by means of feedback, the effect of the disturbance from the output of a given input/state/output dynamical system. In this chapter, we study the disturbance decoupling problem for a simple class of piecewise affine systems, namely piecewise linear systems with only two modes.

In the context of hybrid dynamical systems, the results on the disturbance decoupling problem are limited to jumping hybrid systems [Conte et al., 2015] and switched linear systems [Conte et al., 2014; Otsuka, 2010, 2011, 2015; Yurtseven et al., 2012; Zattoni and Marro, 2013; Zattoni et al., 2016]. In this chapter, we focus on a particular class of hybrid systems exhibiting state-dependent switchings, namely continuous piecewise linear bimodal systems. The main goal of this chapter is to provide necessary and sufficient conditions for the solvability of the disturbance decoupling problem for this class of systems.

The main difference, in the context of disturbance decoupling, between the state-independent and state-dependent switchings stems from the different nature of the set of reachable states by the disturbances for these two cases. In the case of linear state-independent switching systems, the set of states that can be reached from the origin by the disturbances constitute a subspace of the whole state space. In [Otsuka, 2010; Yurtseven et al., 2012], this leads to the solution of the disturbance decoupling problem by following the footsteps of the classical results for the linear systems. However, the same set of states is, in general, neither a subspace nor even a convex set for the case of state-dependent switchings. As such, the ideas employed in the context of linear state-independent switching systems cannot be indiscriminately applied to linear state-dependent switching systems.

To overcome this obstacle, we first investigate under which conditions a given bimodal system is disturbance decoupled. It turns out that one can still provide easily verifiable geometric necessary and sufficient conditions for disturbance decoupling (see Theorem 2.3), even though the set of reachable states does not, in general, enjoy nice geometric properties such as being convex. Based on these geometric necessary and sufficient conditions, we study the disturbance decoupling problem for both state feedback controllers and dynamic feedback controllers. For both feedback schemes, we consider mode-independent and mode-dependent controllers, and provide necessary and sufficient conditions for the solvability of the disturbance decoupling problem. These conditions amount to checking certain subspace inclusions very much analogous to linear systems and linear state-independent switching systems. To verify these conditions, we also propose subspace algorithms.

In the following section, we introduce the class of continuous piecewise linear bimodal systems as well as the disturbance decoupling problem for this class of systems. This is followed by a complete characterization of the disturbance decoupled (open-loop) bimodal systems. Based on this characterization, we first turn our attention to the disturbance decoupling problem by state feedback in Section 2.3. Subsequently, we discuss the disturbance decoupling problem by dynamic feedback in Section 2.4. In order to verify the conditions presented in these sections, we provide subspace algorithms in Section 2.5. Finally, the chapter closes with conclusions in Section 2.6.

2.2 DISTURBANCE DECOUPLED BIMODAL SYSTEMS

We consider bimodal systems of the form

$$\dot{x}(t) = \begin{cases} A_1 x(t) + Ed(t) & \text{if } c^\top x(t) \leqslant 0, \\ A_2 x(t) + Ed(t) & \text{if } c^\top x(t) \geqslant 0, \end{cases} \tag{2.1a}$$

$$z(t) = Hx(t), \tag{2.1b}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $d \in \mathbb{R}^{n_d}$ is the unknown disturbance, $z \in \mathbb{R}^{n_z}$ is the output, and the matrices A_1, A_2, E, H and the vector c are of appropriate sizes. Throughout this chapter we assume that the right-hand side of (2.1a) is continuous in x. In other words, the implication

$$c^\top x = 0 \quad \Rightarrow \quad A_1 x = A_2 x$$

holds. Equivalently, we have

$$A_1 - A_2 = hc^\top \tag{2.2}$$

for a vector $h \in \mathbb{R}^{n_x}$. As such, the right-hand side of (2.1a) is Lipschitz continuous in the variable x. Therefore, for each initial condition x_0 and locally integrable disturbance d there exists a unique absolutely continuous function $x^{x_0,d}(t)$ satisfying $x^{x_0,d}(0) = x_0$ and (2.1a) for almost all t. We denote the corresponding output of the system by $z^{x_0,d}(t)$.

Example 2.1 As an example, consider the bimodal system

$$\dot{x}(t) = \begin{cases} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} d(t) & \text{if } \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \leqslant 0, \\ \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} d(t) & \text{if } \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \geqslant 0, \end{cases}$$

$$z(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t).$$

We use the following definition of disturbance decoupledness.

Definition 2.2 A continuous piecewise linear bimodal system of the form (2.1) is *disturbance decoupled* if

$$z^{x_0,d_1}(t) = z^{x_0,d_2}(t), \qquad \forall t \geqslant 0$$

for all initial states $x_0 \in \mathbb{R}^{n_x}$ and all locally integrable disturbances d_1 and d_2.

In order to find necessary and sufficient conditions for a bimodal system to be disturbance decoupled, we define the set

$$R(x_0, T) := \{x^{x_0,d}(T) \mid d \text{ is locally integrable}\}$$

for each initial state $x_0 \in \mathbb{R}^{n_x}$ and $T \geqslant 0$. It follows immediately that system (2.1) is disturbance decoupled if and only if for every $x_0 \in \mathbb{R}^{n_x}$ and $T \geqslant 0$ the difference between any two elements in $R(x_0, T)$ is in the kernel of H, or equivalently,

$$\bigcup_{T \geqslant 0} \bigcup_{x_0 \in \mathbb{R}^{n_x}} \big(R(x_0, T) + (-R(x_0, T))\big) \subseteq \ker H. \tag{2.3}$$

Neither the set $R(x_0, T)$ nor $R(x_0, T) + (-R(x_0, T))$ is necessarily convex in general. As such, condition (2.3) is rather hard to check. However, by making use of controllable subspaces, as defined in equation (1.2), we can provide an equivalent geometric condition which is easier to verify.

Theorem 2.3 *The system (2.1) is disturbance decoupled if and only if*

$$\langle A_1 \mid \operatorname{im} E\rangle + \langle A_2 \mid \operatorname{im} E\rangle \subseteq \ker H. \tag{2.4}$$

Before we give a proof of Theorem 2.3, we state and prove the following three auxiliary lemmas.

Lemma 2.4 *Let A_1 and A_2 be two square matrices such that $A_1 - A_2 = hc^\top$. Then the rational vector $c^\top(sI - A_1)^{-1}E$ is identically zero if and only if so is $c^\top(sI - A_2)^{-1}E$.*

Proof. We use the well-known identity

$$(sI - X)^{-1} - (sI - Y)^{-1} = (sI - X)^{-1}(X - Y)(sI - Y)^{-1},$$

with $X = A_1$ and $Y = A_2$. By premultiplying both sides by $c^\top$, post-multiplying by E, and using $A_1 - A_2 = hc^\top$ we get

$$c^\top(sI - A_1)^{-1}E - c^\top(sI - A_2)^{-1}E = \\ c^\top(sI - A_1)^{-1}hc^\top(sI - A_2)^{-1}E.$$

Hence, if $c^\top(sI - A_2)^{-1}E$ is identically zero, then so is $c^\top(sI - A_1)^{-1}E$. By symmetry, the converse also holds. ■

Lemma 2.5 *Let A_1 and A_2 be two square matrices such that $A_1 - A_2 = hc^\mathsf{T}$. Then the subspace $\langle A_1 \mid \operatorname{im} E\rangle + \langle A_2 \mid \operatorname{im} E\rangle$ is the smallest subspace containing $\operatorname{im} E$ that is invariant under both A_1 and A_2. Furthermore, if $c^\mathsf{T}(sI - A_1)^{-1}E$ is not identically zero, then*

$$\langle A_1 \mid \operatorname{im} E\rangle + \langle A_2 \mid \operatorname{im} E\rangle = \langle A_1 \mid \operatorname{im} \begin{bmatrix} h & E \end{bmatrix}\rangle. \tag{2.5}$$

Proof. Let $\mathcal{V} = \langle A_1 \mid \operatorname{im} E\rangle + \langle A_2 \mid \operatorname{im} E\rangle$ and $\mathcal{U} = \langle A_1 \mid \operatorname{im} \begin{bmatrix} h & E \end{bmatrix}\rangle$. The subspace $\mathcal{U}$ contains $\operatorname{im} h$ and is invariant under A_1, hence it is also invariant under $A_2 = A_1 - hc^\mathsf{T}$. Since $\mathcal{U}$ contains $\operatorname{im} E$ and $\langle A_i \mid \operatorname{im} E\rangle$ is the smallest A_i-invariant subspace containing $\operatorname{im} E$, we have $\langle A_i \mid \operatorname{im} E\rangle \subseteq \mathcal{U}$ for $i = 1, 2$. Hence, the inclusion $\mathcal{V} \subseteq \mathcal{U}$ follows.

Suppose that

$$c^\mathsf{T}(sI - A_1)^{-1}E = \sum_{k=0}^{\infty} \frac{1}{s^{k+1}} c^\mathsf{T} A_1^k E$$

is not identically zero, and let p be the smallest non-negative integer such that $c^\mathsf{T} A_1^p E \neq 0$. From equation (1.2) it follows that for any element $y \in \mathcal{V}^\perp$ it holds that

$$y^\mathsf{T} A_1^k E = y^\mathsf{T} A_2^k E = 0, \quad \forall k \geqslant 0.$$

In particular, by choosing $k = p + 1$ we obtain

$$0 = y^\mathsf{T} A_2^{p+1} E = y^\mathsf{T}(A_1 - hc^\mathsf{T})^{p+1} E = -y^\mathsf{T} h c^\mathsf{T} A_1^p E,$$

where we use that $c^\mathsf{T} A_1^k E = 0$ for $0 \leqslant k \leqslant p-1$. Since $c^\mathsf{T} A_1^p E$ is nonzero, this implies that $y^\mathsf{T} h = 0$. Hence, we get $h \in (\mathcal{V}^\perp)^\perp = \mathcal{V}$. Consequently, for all $v_1 \in \langle A_1 \mid \operatorname{im} E\rangle$ and $v_2 \in \langle A_2 \mid \operatorname{im} E\rangle$ we have $A_1(v_1 + v_2) = A_1 v_1 + A_2 v_2 + hc^\mathsf{T} v_2 \in \mathcal{V}$. As such, $\mathcal{V}$ is A_1-invariant. Furthermore, $\mathcal{V}$ contains both $\operatorname{im} h$ and $\operatorname{im} E$. It follows that $\mathcal{U} \subseteq \mathcal{V}$, since $\mathcal{U}$ is the smallest A_1-invariant subspace containing $\operatorname{im} h$ and $\operatorname{im} E$. Hence, (2.5) holds. Since $\mathcal{U}$ is invariant under both A_1 and A_2, so is the subspace $\mathcal{V}$.

In the case that $c^\mathsf{T}(sI - A_1)^{-1}E$ is identically zero, we have $c^\mathsf{T} A_1^k E = 0$ for all integers $k \geqslant 0$. We claim that $A_1^k E = A_2^k E$ for all $k \geqslant 0$. To prove this claim, we employ mathematical induction on k. It clearly holds for $k = 0$. Suppose that $A_1^k E = A_2^k E$ holds for $k = 0, 1, \ldots, \ell$, then

$$A_1^{\ell+1} E = A_1 A_1^\ell E = (A_2 + hc^\mathsf{T}) A_1^\ell E = A_2 A_1^\ell E = A_2^{\ell+1} E.$$

Hence, we have $\langle A_1 \mid \operatorname{im} E\rangle = \langle A_2 \mid \operatorname{im} E\rangle = \mathcal{V}$. Consequently, also in this case $\mathcal{V}$ is invariant under both A_1 and A_2.

Since any subspace that contains $\operatorname{im} E$ and is invariant under A_1 and A_2 must contain both $\langle A_1 \mid \operatorname{im} E\rangle$ and $\langle A_2 \mid \operatorname{im} E\rangle$, we see that $\mathcal{V}$ is the smallest of such subspaces. ■

Lemma 2.6 *If $c^\top(sI - A_1)^{-1}E$ is identically zero, then for all initial states $x_0 \in \mathbb{R}^n$ and integrable disturbances d_1 and d_2 we have $c^\top x^{x_0,d_1}(t) = c^\top x^{x_0,d_2}(t)$ for all $t \geqslant 0$.*

Proof. Let $\mathcal{V} = \langle A_1 \mid \operatorname{im} E\rangle + \langle A_2 \mid \operatorname{im} E\rangle$. It follows from Lemma 2.4 that $c^\top A_1^k E = c^\top A_2^k E = 0$ for $k \geqslant 0$. Hence, we get $\mathcal{V} \subseteq \ker c^\top$. By Lemma 2.5, $\mathcal{V}$ is invariant under both A_1 and A_2. Let $v_1, v_2, \dots, v_\ell$ be a basis for $\mathcal{V}$, and extend this to a basis $v_1, v_2, \dots, v_{n_x}$ for $\mathbb{R}^{n_x}$. Let $S = \begin{bmatrix} v_1 & v_2 & \dots & v_{n_x}\end{bmatrix}$, then the basis transformation $\xi = S^{-1}x$ results in the system

$$\dot{\xi}(t) = \begin{cases} \tilde{A}_1\xi(t) + \tilde{E}d(t) & \text{if } \tilde{c}^\top\xi(t) \leqslant 0, \\ \tilde{A}_2\xi(t) + \tilde{E}d(t) & \text{if } \tilde{c}^\top\xi(t) \geqslant 0. \end{cases}$$

Decompose ξ as $\xi = \operatorname{col}(\xi_1, \xi_2)$, where ξ_1 contains the first ℓ entries of ξ. Since $\operatorname{im} E \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \ker c^\top$, we see that the matrices $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{E}$ and $\tilde{c}^\top$ are of the form

$$\tilde{A}_1 = \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ 0 & A_{22}^1 \end{bmatrix}, \qquad \tilde{A}_2 = \begin{bmatrix} A_{11}^2 & A_{12}^2 \\ 0 & A_{22}^2 \end{bmatrix},$$
$$\tilde{E} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \qquad \tilde{c}^\top = \begin{bmatrix} 0 & c_2^\top \end{bmatrix}.$$

In particular, ξ_2 satisfies

$$\dot{\xi}_2 = \begin{cases} A_{22}^1\xi_2 & \text{if } c_2^\top\xi_2 \leqslant 0, \\ A_{22}^2\xi_2 & \text{if } c_2^\top\xi_2 \geqslant 0. \end{cases}$$

Note that ξ_2 does not depend on the disturbance d. Therefore, the value of $c^\top x = \tilde{c}^\top\xi = c_2^\top\xi_2$ does not depend on the disturbance. Hence, we see that $c^\top x^{x_0,d_1}(t) = c^\top x^{x_0,d_2}(t)$ for all $t \geqslant 0$, initial conditions x_0 and integrable disturbances d_1, d_2. ■

Now we are in a position to give a proof of Theorem 2.3.

Proof of Theorem 2.3. *Necessity:* Suppose that the system (2.1) is disturbance decoupled. Let x_0 be such that $c^\top x_0 < 0$ and take $d_1(t) = d \in \mathbb{R}^{n_d}$ a constant vector, and $d_2(t) = 0$. Let $x_i(t) = x^{x_0,d_i}(t)$ for $i = 1, 2$ denote the state trajectories of the system (2.1) corresponding to the initial state x_0 and disturbances d_i, and let $z_i(t) = Hx_i(t)$ denote their outputs. Since x_1 and x_2 are continuous, there exists an $\varepsilon > 0$ such that $c^\top x_i(t) < 0$ for all $t \in (0, \varepsilon)$ and $i = 1, 2$. This means that for $t \in (0, \varepsilon)$ we have

$$\dot{x}_i(t) = A_1 x_i(t) + E d_i(t), \qquad i = 1, 2.$$

Since the system (2.1) is disturbance decoupled, the outputs satisfy $z_1(t) = z_2(t)$ for $t \geqslant 0$. Therefore, we have

$$H x_1(t) = H x_2(t), \qquad t \geqslant 0.$$

Note that $d_1(t)$ and $d_2(t)$ are both taken to be constant, so we can differentiate both sides k times, resulting in

$$H A_1^k x_1(t) + H A_1^{k-1} E d = H A_1^k x_2(t), \quad t \in (0, \varepsilon), \quad k \geqslant 1.$$

Taking the limit as $t \downarrow 0$ and using $x_1(0) = x_2(0)$ gives us

$$H A_1^k E d = 0, \qquad k \geqslant 0.$$

Since this holds for every vector $d \in \mathbb{R}^q$, we can conclude that $H A_1^k E = 0$ for all $k \geqslant 0$. By choosing x_0 such that $c^\top x_0 > 0$ and employing similar arguments, we obtain $H A_2^k E = 0$ for all $k \geqslant 0$. Consequently, (2.4) holds.

Sufficiency: Let $\mathcal{V} = \langle A_1 \mid \operatorname{im} E \rangle + \langle A_2 \mid \operatorname{im} E \rangle$. In view of (2.3), it suffices to show that $R(x_0, T) - R(x_0, T) \subseteq \mathcal{V}$, or equivalently $\mathcal{V}^\perp \subseteq (R(x_0, T) - R(x_0, T))^\perp$ for all x_0 and $T \geqslant 0$.

Let x_0 be an initial state and d_1, d_2 two disturbances. Also, let $x_i(t) = x^{x_0, d_i}(t)$ for $i = 1, 2$ denote the two corresponding trajectories of the system (2.1). Let y be an element of $\mathcal{V}^\perp = \langle A_1 \mid \operatorname{im} E \rangle^\perp \cap \langle A_2 \mid \operatorname{im} E \rangle^\perp$. Then $y^\top A_1^k E = 0$ and $y^\top A_2^k E = 0$ for all $k \geqslant 0$. In the case that $c^\top (sI - A_1)^{-1} E$ is not identically zero, we know from Lemma 2.5 that $\operatorname{im} h \subseteq \mathcal{V}$. As such, we have $y^\top h = 0$. Together with $y^\top E = 0$, this yields

$$\begin{aligned} y^\top \dot{x}_i(t) &= \begin{cases} y^\top A_1 x_i(t) & \text{if } c^\top x_i(t) \leqslant 0 \\ y^\top A_2 x_i(t) & \text{if } c^\top x_i(t) \geqslant 0 \end{cases} \\ &= y^\top A_1 x_i(t), \end{aligned}$$

for $t \geqslant 0$ and $i = 1, 2$. In the case that $c^\top (sI - A_1)^{-1} E$ is identically zero, it follows from Lemma 2.6 that $c^\top x_1(t) =$

$c^\top x_2(t)$ for all $t \geqslant 0$. Hence, we have $hc^\top(x_1(t) - x_2(t)) = 0$, which implies that

$$y^\top(\dot{x}_1(t) - \dot{x}_2(t)) = \begin{cases} y^\top A_1(x_1(t) - x_2(t)), & c^\top x_1(t) \leqslant 0 \\ y^\top A_2(x_1(t) - x_2(t)), & c^\top x_1(t) \geqslant 0 \end{cases}$$
$$= y^\top A_1(x_1(t) - x_2(t)),$$

for $t \geqslant 0$.

In conclusion, in both cases we have

$$y^\top(\dot{x}_1(t) - \dot{x}_2(t)) = y^\top A_1(x_1(t) - x_2(t)), \tag{2.6}$$

for all $y \in \mathcal{V}^\perp$ and for almost all $t \geqslant 0$. To study equation (2.6), we first suppose that λ is an eigenvalue of $A_1^\top$ and $y \in \mathcal{V}^\perp$ satisfies

$$(A_1^\top - \lambda I)^k y = 0 \tag{2.7}$$

for some integer $k \geqslant 1$. The vector y generates a Jordan chain $y_1, y_2, \ldots, y_k$ for the eigenvalue λ as follows:

$$y_j = (A_1^\top - \lambda I)^{k-j} y \quad \text{for} \quad 1 \leqslant j \leqslant k.$$

Since $y_k = y \in \mathcal{V}^\perp$ and $\mathcal{V}^\perp$ is $A_1^\top$-invariant, we see that $y_j \in \mathcal{V}^\perp$ for all $j = 1, 2, \ldots, k$. We will prove by mathematical induction on j that

$$y_j^\top(x_1(t) - x_2(t)) = 0 \tag{2.8}$$

for $j = 1, 2, \ldots, k$ and all $t \geqslant 0$. For $j = 1$, we have $A_1^\top y_1 = \lambda y_1$. Hence, it follows from (2.6) that

$$\frac{d}{dt}[y_1^\top(x_1(t) - x_2(t))] = \lambda y_1^\top(x_1(t) - x_2(t)),$$

for almost all $t \geqslant 0$. This results in

$$y_1^\top(x_1(t) - x_2(t)) = e^{\lambda t} y_1^\top(x_1(0) - x_2(0)) = 0,$$

since $x_1(0) = x_2(0)$. Now, assume that (2.8) holds for $j = 1, 2, \ldots, \ell$ for some integer ℓ with $1 \leqslant \ell < k$. By using (2.6) and $A_1^\top y_{\ell+1} = \lambda y_{\ell+1} + y_\ell$, we obtain

$$\begin{aligned} \frac{d}{dt}[y_{\ell+1}^\top(x_1(t) - x_2(t))] &= y_{\ell+1}^\top A_1(x_1(t) - x_2(t)) \\ &= (\lambda y_{\ell+1} + y_\ell)^\top(x_1(t) - x_2(t)) \\ &= \lambda y_{\ell+1}^\top(x_1(t) - x_2(t)), \end{aligned}$$

for almost all $t \geqslant 0$. Therefore, we have

$$y_{\ell+1}^{\top}(x_1(t) - x_2(t)) = e^{\lambda t} y_{\ell+1}^{\top}(x_1(0) - x_2(0)) = 0.$$

This completes the proof of (2.8). Clearly, (2.8) implies that $y_j \in (R(x_0, T) - R(x_0, T))^{\perp}$ for all j, x_0 and $T \geqslant 0$.

To generalize this result to all $y \in \mathcal{V}^{\perp}$, we define $\mathcal{M} \subseteq \mathbb{C}^n$ to be the subspace $\mathcal{M} = \mathcal{V}^{\perp} \oplus i\mathcal{V}^{\perp}$. Consider $A_1^{\top}$ as a linear map from $\mathbb{C}^n$ to $\mathbb{C}^n$. Since $\mathcal{V}^{\perp}$ is $A_1^{\top}$-invariant, so is $\mathcal{M}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the distinct eigenvalues of $A_1^{\top}$ and define the corresponding root subspaces R_{λ_i} for $i = 1, 2, \ldots, r$ as

$$\mathcal{R}_{\lambda_i}(A_1^{\top}) := \ker(A_1^{\top} - \lambda_i I)^{p_i},$$

where p_i is the geometric multiplicity of the eigenvalue λ_i. By [Gohberg et al., 2006, Thm. 2.1.5], we can decompose $\mathcal{M}$ as follows:

$$\mathcal{M} = \bigoplus_{i=1}^{r} (\mathcal{M} \cap \mathcal{R}_{\lambda_i}(A_1^{\top})).$$

For fixed x_0 and $T \geqslant 0$, we can consider $R(x_0, T) - R(x_0, T)$ as a subset of $\mathbb{C}^n$. Since each root subspace R_{λ_i} is spanned by a Jordan chain, it follows from the preceding argument on Jordan chains that $\mathcal{M} \subseteq (R(x_0, T) - R(x_0, T))^{\perp}$. Hence, $\mathcal{V}^{\perp} \subseteq (R(x_0, T) - R(x_0, T))^{\perp}$ for all x_0 and $T \geqslant 0$, which completes the proof. ∎

For later use in the next two sections, and to relate our result to similar results for switched linear systems, we state the following corollary, which follows from combining Theorem 2.3 with Lemma 2.5.

Corollary 2.7 *The system (2.1) is disturbance decoupled if and only if there exists a subspace $\mathcal{V} \subseteq \mathbb{R}^{n_x}$ that is invariant under both A_1 and A_2 such that $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$.*

Example 2.8 We revisit Example 2.1. For this system, notice that $A_1 E = E$ and $A_2 E = E$, which implies that $\langle A_1 \mid \operatorname{im} E \rangle = \langle A_2 \mid \operatorname{im} E \rangle = \operatorname{im} E$. Since $HE = 0$, we see that this bimodal system satisfies (2.4), and hence it is disturbance decoupled.

Remark 2.9 In [Yurtseven et al., 2012], the disturbance decoupling problem for switched linear systems is studied. The results presented in [Yurtseven et al., 2012, Thm. 3.7 and 3.9] provide sufficient conditions for the disturbance decoupling of piecewise linear systems. Applied to the bimodal system (2.1), these conditions boil down to the conditions in Corollary 2.7, but with the extra condition that the subspace $\mathcal{V}$ and the matrices A_1 and A_2 should satisfy $\operatorname{im}(A_1 - A_2) = \operatorname{im} hc^\top \subseteq \mathcal{V}$. This last condition implies that $h \in \mathcal{V}$, which is not necessary in the case that $c^\top(sI - A_1)^{-1}E$ is identically zero.

2.3 DISTURBANCE DECOUPLING BY STATE FEEDBACK

The next question we address is under what conditions a bimodal system can be rendered disturbance decoupled by means of static state feedback. To do so, we consider the bimodal system

$$\dot{x}(t) = \begin{cases} A_1x(t) + Bu(t) + Ed(t) & \text{if } c^\top x(t) \leqslant 0 \\ A_2x(t) + Bu(t) + Ed(t) & \text{if } c^\top x(t) \geqslant 0 \end{cases} \tag{2.9a}$$

$$z(t) = Hx(t) \tag{2.9b}$$

where $u \in \mathbb{R}^{n_u}$ is the input, B is an $n_x \times n_u$ input matrix, and x, z, d, A_1, A_2, E, H and c are as before. We assume that B has full column rank and that A_1 and A_2 satisfy the continuity condition (2.2).

In this section we provide necessary and sufficient conditions for the existence of a static state feedback law that renders the closed-loop system disturbance decoupled. We consider two forms of static feedback: mode-dependent and mode-independent.

2.3.1 *Mode-dependent state feedback*

Consider a mode-dependent static feedback law of the form

$$u(t) = \begin{cases} F_1x(t) & \text{if } c^\top x(t) \leqslant 0 \\ F_2x(t) & \text{if } c^\top x(t) \geqslant 0 \end{cases} \tag{2.10}$$

for two matrices $F_1, F_2 \in \mathbb{R}^{n_u \times n_x}$ with the property that $c^\top x = 0$ implies $F_1x = F_2x$, or equivalently, $\ker c^\top \subseteq \ker(F_1 - F_2)$. This implies that there exists a vector $f \in \mathbb{R}^{n_u}$ such that

$$F_1 - F_2 = fc^\top. \tag{2.11}$$

In other words, we consider mode-dependent and continuous (in x) state feedback. Clearly, such a feedback results in the (continuous) closed-loop system

$$\dot{x}(t) = \begin{cases} (A_1 + BF_1)x(t) + Ed(t) & \text{if } c^\top x(t) \leqslant 0 \\ (A_2 + BF_2)x(t) + Ed(t) & \text{if } c^\top x(t) \geqslant 0 \end{cases} \tag{2.12a}$$

$$z(t) = Hx(t). \tag{2.12b}$$

In view of Corollary 2.7, we see that the closed-loop system (2.12) is disturbance decoupled if and only if there exist a subspace $\mathcal{V}$ and matrices F_1 and F_2 such that $\mathcal{V}$ is invariant under both $A_1 + BF_1$ and $A_2 + BF_2$, $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$, and $\ker c^\top \subseteq \ker(F_1 - F_2)$.

In order to check whether such a subspace exists or not, we need to introduce some nomenclature. Define the set of subspaces

$$\begin{aligned} &V_{\mathrm{md}}(H, \{A_1, A_2\}, B) := \\ &\quad \{\, \mathcal{V} \subseteq \ker H \mid \exists F_1, F_2 \text{ s.t. } (A_j + BF_j)\mathcal{V} \subseteq \mathcal{V},\ j = 1, 2 \,\}, \end{aligned} \tag{2.13}$$

where the subscript 'md' stands for mode-dependent. Let $\mathcal{V}$ and $\mathcal{W}$ be two subspaces in $V_{\mathrm{md}}(H, \{A_1, A_2\}, B)$. Then, since $\mathcal{V}$ and $\mathcal{W}$ are both (A_1, B)-invariant, the subspace $\mathcal{V} + \mathcal{W}$ is (A_1, B)-invariant as well. Similarly, we see that $\mathcal{V} + \mathcal{W}$ is (A_2, B)-invariant too. Therefore, $V_{\mathrm{md}}(H, \{A_1, A_2\}, B)$ is closed under subspace addition. Let $\mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B)$ be the largest of the subspaces in $V_{\mathrm{md}}(H, \{A_1, A_2\}, B)$. If the context is clear, we will denote it by $\mathcal{V}^*_{\mathrm{md}}$.

Note that in the definition of $V_{\mathrm{md}}(H, \{A_1, A_2\}, B)$ in (2.13) we do not consider the continuity condition (2.11). However, for any subspace $\mathcal{V}$ in $V_{\mathrm{md}}(\ker H, \{A_1, A_2\}, B)$, there exist matrices F_1 and F_2 such that the feedback (2.10) is continuous in x and $(A_i + BF_i)\mathcal{V} \subseteq \mathcal{V}$ for $i = 1, 2$, as shown in the following lemma.

Lemma 2.10 *If a subspace $\mathcal{V}$ is (A_1, B)-invariant and (A_2, B)-invariant, and $A_1 - A_2 = hc^\top$, then there exist matrices $F_1, F_2 \in \mathbb{R}^{n_u \times n_x}$ such that $F_1 - F_2 = fc^\top$ for some $f \in \mathbb{R}^{n_u}$ and $(A_i + BF_i)\mathcal{V} \subseteq \mathcal{V}$ for $i = 1, 2$.*

Proof. $\mathcal{V}$ is (A_1, B)-invariant, so there exists a matrix F_1 such that $(A_1 + BF_1)\mathcal{V} \subseteq \mathcal{V}$. Since $\mathcal{V}$ is (A_2, B)-invariant as well, $\mathcal{V}$ is also $(hc^\top, B)$-invariant, so $hc^\top \mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$. This implies that we have $h \in \mathcal{V} + \operatorname{im} B$ or $\mathcal{V} \subseteq \ker c^\top$. In the former case, there exists an $f \in \mathbb{R}^{n_u}$ such that $h + Bf \in \mathcal{V}$. In the latter case, let f be any vector in $\mathbb{R}^{n_u}$. Hence, in both cases we have $(h + Bf)c^\top \mathcal{V} \subseteq \mathcal{V}$. Let $F_2 = F_1 - fc^\top$, then $A_2 + BF_2 = A_1 + BF_1 - (h + Bf)c^\top$, which implies that $(A_2 + BF_2)\mathcal{V} \subseteq \mathcal{V}$. ∎

The following theorem shows how we can use the subspace $\mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B)$ to determine whether there exists a mode-dependent state feedback controller that renders the system (2.9) disturbance decoupled.

Theorem 2.11 *There exists a mode-dependent static state feedback of the form* (2.10) *that renders the closed-loop system (2.12) disturbance decoupled if and only if*

$$\operatorname{im} E \subseteq \mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B).$$

Proof. *Sufficiency:* Since $\mathcal{V}^*_{\mathrm{md}}$ is (A_1, B)-invariant and (A_2, B)-invariant, by Lemma 2.10 there exist matrices F_1 and F_2 such that $F_1 - F_2 = fc^\top$ for some $f \in \mathbb{R}^{n_u}$ and $(A_i + BF_i)\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}^*_{\mathrm{md}}$ for $i = 1, 2$. From the hypothesis, we have $\operatorname{im} E \subseteq \mathcal{V}^*_{\mathrm{md}} \subseteq \ker H$. Then, it follows from Corollary 2.7 that mode-dependent static feedback given by (2.10) renders the closed-loop system (2.12) disturbance decoupled.

Necessity: Suppose that F_1 and F_2 are such that the input (2.10) renders the closed-loop system (2.12) disturbance decoupled. It follows from Corollary 2.7 that there exists a subspace $\mathcal{V}$ that is invariant under both $A_1 + BF_1$ and $A_2 + BF_2$, and such that $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$. Therefore, $\mathcal{V} \in V_{\mathrm{md}}(H, \{A_1, A_2\}, B)$. Hence, $\operatorname{im} E \subseteq \mathcal{V} \subseteq \mathcal{V}^*_{\mathrm{md}}$. ∎

In Section 2.5 we will provide an algorithm to compute the subspace $\mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B)$. Once the condition $\operatorname{im} E \subseteq \mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B)$ is satisfied, one can construct the feedback matrices F_1 and F_2 by following the steps in the proof of Lemma 2.10.

2.3.2 *Mode-independent state feedback*

Consider the static state feedback law $u = Fx$ for a matrix $F \in \mathbb{R}^{n_u \times n_x}$. This can be seen as a special case of the mode-dependent state feedback, with $F_1 = F_2$. Such a feedback law results in the closed-loop system

$$\dot{x}(t) = \begin{cases} (A_1 + BF)x(t) + Ed(t) & \text{if } c^\top x(t) \leqslant 0 \\ (A_2 + BF)x(t) + Ed(t) & \text{if } c^\top x(t) \geqslant 0. \end{cases} \tag{2.14}$$

By Corollary 2.7, we see that the closed-loop system is disturbance decoupled if and only if there exist a subspace $\mathcal{V}$ and a feedback matrix F such that $\mathcal{V}$ is invariant under both $A_1 + BF$ and $A_2 + BF$, and $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$. Similar to the mode-dependent case, we define the set of subspaces

$$\begin{aligned} &V_{\mathrm{mi}}(H, \{A_1, A_2\}, B) := \\ &\quad \{\, \mathcal{V} \subseteq \ker H \mid \exists F \text{ s.t. } (A_j + BF)\mathcal{V} \subseteq \mathcal{V} \text{ for } j = 1, 2 \,\}, \end{aligned} \tag{2.15}$$

where the subscript 'mi' stands for mode-independent. The set $V_{\mathrm{mi}}(H, \{A_1, A_2\}, B)$ is closed under subspace addition, and hence it has a largest element, denoted by $\mathcal{V}^*_{\mathrm{mi}}(H, \{A_1, A_2\}, B)$, or simply by $\mathcal{V}^*_{\mathrm{mi}}$ if the context is clear. In Section 2.5 we provide an algorithm to compute $\mathcal{V}^*_{\mathrm{mi}}$.

The following theorem can be proven by using similar arguments as employed in the proof of Theorem 2.11.

Theorem 2.12 *There exists a matrix $F \in \mathbb{R}^{n_u \times n_x}$ such that the state feedback $u(t) = Fx(t)$ renders the closed-loop system (2.9) disturbance decoupled if and only if*

$$\operatorname{im} E \subseteq \mathcal{V}^*_{\mathrm{mi}}(H, \{A_1, A_2\}, B).$$

2.4 DISTURBANCE DECOUPLING BY DYNAMIC FEEDBACK

In this section, we address the disturbance decoupling problem by dynamic feedback. Consider the bimodal system (2.9) together with the output

$$y(t) = Cx(t), \tag{2.16}$$

where $y \in \mathbb{R}^{n_y}$. The main goal of this section is to investigate under which conditions there exists a dynamic controller from y to u rendering the closed-loop system disturbance decoupled. Similar to the state feedback problem, we distinguish two cases: mode-dependent and mode-independent controllers.

2.4.1 *Mode-dependent dynamic feedback*

We start with the mode-dependent dynamic feedback controller given by

$$\dot{w}(t) = \begin{cases} Kw(t) + L_1 y(t) & \text{if } c^\top x \leqslant 0 \\ Kw(t) + L_2 y(t) & \text{if } c^\top x \geqslant 0 \end{cases} \tag{2.17a}$$

$$u(t) = \begin{cases} Mw(t) + N_1 y(t) & \text{if } c^\top x \leqslant 0 \\ Mw(t) + N_2 y(t) & \text{if } c^\top x \geqslant 0 \end{cases} \tag{2.17b}$$

where $w \in \mathbb{R}^{n_w}$ is the state of the controller, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ are as before, and the matrices K, L_1, L_2, M, N_1 and N_2 are of suitable sizes. Interconnecting this controller with the system given by (2.9) and (2.16) results in the closed-loop system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{cases} A_{e,1} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + E_e d(t) & \text{if } c_e^\top \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \leqslant 0 \\ \\ A_{e,2} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + E_e d(t) & \text{if } c_e^\top \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \geqslant 0 \end{cases} \tag{2.18a}$$

$$z(t) = H_e \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \tag{2.18b}$$

where

$$A_{e,i} = \begin{bmatrix} A_i + BN_iC & BM \\ L_iC & K \end{bmatrix}, \qquad i = 1, 2, \tag{2.18c}$$

$$E_e = \begin{bmatrix} E \\ 0 \end{bmatrix}, \quad H_e = \begin{bmatrix} H & 0 \end{bmatrix}, \quad c_e^\top = \begin{bmatrix} c^\top & 0 \end{bmatrix}. \tag{2.18d}$$

We only consider mode-dependent feedback controllers that render the closed-loop system continuous (both in x and w). This amounts to imposing the following conditions on the matrices L_1, L_2, N_1 and N_2:

$$\ker c^\top \subseteq \ker(L_1 - L_2)C, \quad \ker c^\top \subseteq \ker(N_1 - N_2)C. \tag{2.19}$$

Equivalently, we assume that there are vectors $\ell \in \mathbb{R}^{n_w}$ and $n \in \mathbb{R}^{n_u}$ such that

$$(L_1 - L_2)C = \ell c^\top, \quad (N_1 - N_2)C = nc^\top. \tag{2.20}$$

As a result, we have $\ker c_e^\top \subseteq \ker(A_{e,1} - A_{e,2})$.

The objective of this section is to find such a mode-dependent dynamic controller that renders the closed-loop system disturbance decoupled. By employing (C, A_1, B)-pairs (see Section 1.5.1), the following theorem provides necessary and sufficient conditions for the existence of such a controller.

Theorem 2.13 *There exists a mode-dependent dynamic controller of the form* (2.17) *satisfying the continuity condition* (2.19) *such that the closed-loop system* (2.18) *is disturbance decoupled if and only if there exist subspaces $\mathcal{T}$ and $\mathcal{V}$ such that $(\mathcal{T}, \mathcal{V})$ is a (C, A_1, B)-pair satisfying* $\operatorname{im} E \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \ker H$ *and* $hc^\top \mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$.

Proof. *Necessity:* Assume that there exists such a controller given by K, L_1, L_2, M, N_1 and N_2. Let $\mathbb{R}^{n_w}$ denote the state space of the controller. The (extended) state space of the interconnected system is then given by $\mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$. By Corollary 2.7, there exists a subspace $\mathcal{V}_\mathrm{e} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$ that is invariant under both $A_{\mathrm{e},1}$ and $A_{\mathrm{e},2}$, satisfying $\operatorname{im} E_\mathrm{e} \subseteq \mathcal{V}_\mathrm{e} \subseteq \ker H_\mathrm{e}$. For this subspace $\mathcal{V}_\mathrm{e}$, we define the following two subspaces of $\mathbb{R}^{n_x}$:

$$p(\mathcal{V}_\mathrm{e}) := \{x \in \mathbb{R}^{n_x} \mid \exists w \in \mathbb{R}^{n_w} \text{ such that } \begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{V}_\mathrm{e}\},$$

$$i(\mathcal{V}_\mathrm{e}) := \{x \in \mathbb{R}^{n_x} \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{V}_\mathrm{e}\},$$

which can be seen as the projection of $\mathcal{V}_\mathrm{e}$ on $\mathbb{R}^{n_x}$ and the intersection of $\mathcal{V}_\mathrm{e}$ and $\mathbb{R}^{n_x} \times \{0\}$ respectively. Let $\mathcal{T} = i(\mathcal{V}_\mathrm{e})$ and $\mathcal{V} = p(\mathcal{V}_\mathrm{e})$. Since $\mathcal{V}_\mathrm{e}$ is $A_{\mathrm{e},1}$-invariant, $(\mathcal{T}, \mathcal{V})$ is a (C, A_1, B)-pair (see e.g. [Trentelman et al., 2001, Theorem 6.2]). Next, we will show that this (C, A_1, B)-pair $(\mathcal{T}, \mathcal{V})$ satisfies $\operatorname{im} E \subseteq \mathcal{T}$, $\mathcal{V} \subseteq \ker H$ and $hc^\top \mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$.

For any $x \in \operatorname{im} E$, we have that $\operatorname{col}(x, 0) \in \operatorname{im} E_\mathrm{e} \subseteq \mathcal{V}_\mathrm{e}$. Therefore, we get $x \in i(\mathcal{V}_\mathrm{e}) = \mathcal{T}$, hence we have $\operatorname{im} E \subseteq \mathcal{T}$. For $x \in \mathcal{V} = p(\mathcal{V}_\mathrm{e})$, there exists a $w \in \mathbb{R}^{n_x}$ such that $\operatorname{col}(x, w) \in \mathcal{V}_\mathrm{e} \subseteq \ker H_\mathrm{e}$. Then, we get $Hx = H_\mathrm{e} \operatorname{col}(x, w) = 0$ and hence $\mathcal{V} \subseteq \ker H$.

Since L_1, L_2, N_1 and N_2 satisfy (2.19), there are vectors ℓ and n such that (2.20) holds. Consequently, we have

$$A_{\mathrm{e},1} - A_{\mathrm{e},2} = \begin{bmatrix} (h + Bn)c^\top & 0 \\ \ell c^\top & 0 \end{bmatrix}. \tag{2.21}$$

Let $x \in \mathcal{V}$. Then, a $w \in \mathbb{R}^w$ such that $(x^\top, w^\top)^\top \in \mathcal{V}_\mathrm{e}$. Since $\mathcal{V}_\mathrm{e}$ is invariant under both $A_{\mathrm{e},1}$ and $A_{\mathrm{e},2}$, we have

$$(A_{\mathrm{e},1} - A_{\mathrm{e},2}) \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} (h + Bn)c^\top x \\ \ell c^\top x \end{bmatrix} \in \mathcal{V}_\mathrm{e}.$$

Consequently, we obtain $(h+Bn)c^{\mathsf{T}}x \in \mathcal{V}$ and hence $hc^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$.

Sufficiency: Let $(\mathcal{T},\mathcal{V})$ be a such a (C, A_1, B)-pair. Then there exist F and G such that

$$(A_1 + BF)\mathcal{V} \subseteq \mathcal{V}, \qquad (A_1 + GC)\mathcal{T} \subseteq \mathcal{T}.$$

Furthermore, there exists a linear mapping N_1 such that (see e.g. [Trentelman et al., 2001, Lemma 6.3])

$$(A_1 + BN_1C)\mathcal{T} \subseteq \mathcal{V}.$$

Since $hc^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$, we have $\mathcal{V} \subseteq \ker c^{\mathsf{T}}$ or $h \in \mathcal{V} + \operatorname{im} B$. If the latter holds, then there exists an $n \in \mathbb{R}^{n_u}$ such that $h + Bn \in \mathcal{V}$. Then choose $\ell \in \mathcal{V}$ such that $h + Bn - \ell \in \mathcal{T}$. If we have $\mathcal{V} \subseteq \ker c^{\mathsf{T}}$, then we can choose $n \in \mathbb{R}^{n_u}$ and $\ell \in \mathcal{V}$ arbitrarily. In both cases, we can find n and ℓ such that $(h + Bn - \ell)c^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{T}$.

Let $L_1 = BN_1 - G$ and define

$$\begin{aligned} K &= A_1 + BF + GC - BN_1C, &\qquad L_2 &= L_1 - \ell c^{\mathsf{T}}, \\ M &= F - N_1C, & N_2 &= N_1 - nc^{\mathsf{T}}. \end{aligned}$$

and let K, L_1, L_2, M, N_1 and N_2 define a controller of the form (2.17), with $n_w = n_x$. Note that $(L_1 - L_2)C = \ell c^{\mathsf{T}}$ and $(N_1 - N_2)C = nc^{\mathsf{T}}$, so L_1, L_2, N_1 and N_2 satisfy the continuity condition (2.19). The system matrices of the corresponding closed-loop system (2.18) are then given by

$$A_{e,i} = \begin{bmatrix} A_i + BN_iC & B(F - N_1C) \\ L_iC & A_1 + BF + GC - BN_1C \end{bmatrix},$$

for $i = 1, 2$.

Let $\mathcal{V}_e$ be the subspace of $\mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$ given by

$$\mathcal{V}_e = \{ \begin{bmatrix} s \\ 0 \end{bmatrix} + \begin{bmatrix} v \\ v \end{bmatrix} \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \mid s \in \mathcal{T}, v \in \mathcal{V} \}.$$

First we show that $\mathcal{V}_e$ is invariant under both $A_{e,1}$ and $A_{e,2}$. For any $s \in \mathcal{T}$ and $v \in \mathcal{V}$ we have that

$$\begin{aligned} A_{e,1}\begin{bmatrix} s \\ 0 \end{bmatrix} &= \begin{bmatrix} (A_1 + GC)s \\ 0 \end{bmatrix} + \begin{bmatrix} (A_1 + BNC)s \\ (A_1 + BNC)s \end{bmatrix} \\ &\qquad - \begin{bmatrix} (A_1 + GC)s \\ (A_1 + GC)s \end{bmatrix}, \\ A_{e,1}\begin{bmatrix} v \\ v \end{bmatrix} &= \begin{bmatrix} (A_1 + BF)v \\ (A_1 + BF)v \end{bmatrix}, \end{aligned}$$

are both elements of $\mathcal{V}_{\mathrm{e}}$, so $\mathcal{V}_{\mathrm{e}}$ is invariant under $A_{\mathrm{e},1}$. Using equation (2.21), $\ell \in \mathcal{V}$, and $(h+Bn-\ell)c^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{T}$, we see that for all $s \in \mathcal{T}$ and $v \in \mathcal{V}$ we have that

$$\begin{aligned}
(A_{\mathrm{e},1} - A_{\mathrm{e},2})\begin{bmatrix} s \\ 0 \end{bmatrix} &= \begin{bmatrix} (h+Bn)c^{\mathsf{T}}s \\ \ell c^{\mathsf{T}}s \end{bmatrix} \\
&= \begin{bmatrix} \ell c^{\mathsf{T}}s \\ \ell c^{\mathsf{T}}s \end{bmatrix} + \begin{bmatrix} (h+Bn-\ell)c^{\mathsf{T}}s \\ 0 \end{bmatrix}, \\
(A_{\mathrm{e},1} - A_{\mathrm{e},2})\begin{bmatrix} v \\ v \end{bmatrix} &= \begin{bmatrix} (h+Bn)c^{\mathsf{T}}v \\ \ell c^{\mathsf{T}}v \end{bmatrix} \\
&= \begin{bmatrix} \ell c^{\mathsf{T}}v \\ \ell c^{\mathsf{T}}v \end{bmatrix} + \begin{bmatrix} (h+Bn-\ell)c^{\mathsf{T}}v \\ 0 \end{bmatrix},
\end{aligned}$$

are also both elements of $\mathcal{V}_{\mathrm{e}}$. Consequently, $A_{\mathrm{e},2}\operatorname{col}(s,0)$ and $A_{\mathrm{e},2}\operatorname{col}(v,v)$ are contained in $\mathcal{V}_{\mathrm{e}}$ as well. Therefore, $\mathcal{V}_{\mathrm{e}}$ is invariant under both $A_{\mathrm{e},1}$ and $A_{\mathrm{e},2}$.

Next, we show that $\mathcal{V}_{\mathrm{e}}$ satisfies $\operatorname{im} E_{\mathrm{e}} \subseteq \mathcal{V}_{\mathrm{e}} \subseteq \ker H_{\mathrm{e}}$. For any point $\operatorname{col}(x,w) \in \operatorname{im} E_{\mathrm{e}}$, we have $x \in \operatorname{im} E \subseteq \mathcal{T}$ and $w=0$. Consequently, $\operatorname{col}(x,w) = \operatorname{col}(x,0) \in \mathcal{V}_{\mathrm{e}}$ and hence $\operatorname{im} E_{\mathrm{e}} \subseteq \mathcal{V}_{\mathrm{e}}$. Further, we have $x \in \mathcal{V} \subseteq \ker H$ for any $\operatorname{col}(x,w) \in \mathcal{V}_{\mathrm{e}}$. This implies that $H_{\mathrm{e}}\operatorname{col}(x,w) = Hx = 0$, i.e. $\operatorname{col}(x,w) \in \ker H_{\mathrm{e}}$. Then, we can conclude that $\mathcal{V}_{\mathrm{e}} \subseteq \ker H_{\mathrm{e}}$. Now we can use Corollary 2.7 to prove that the closed-loop system (2.18) is disturbance decoupled. ∎

The conditions presented in Theorem 2.13 are existential in nature. Next, we articulate these conditions and provide easily verifiable conditions based on subspace algorithms. Recall that $\mathcal{T}^*(E,A_1,C)$ is the smallest (C,A_1)-invariant subspace containing $\operatorname{im} E$.

Theorem 2.14 *There exists a mode-dependent dynamic controller of the form* (2.17) *satisfying the continuity condition* (2.19) *that renders the closed-loop system* (2.18) *disturbance decoupled if and only if*

$$\mathcal{T}^*(E,A_1,C) \subseteq \mathcal{V}^*_{\mathrm{md}}(H,\{A_1,A_2\},B).$$

Proof. *Necessity*: If there exists such a controller, then by Theorem 2.13 there are subspaces $\mathcal{T}$ and $\mathcal{V}$ such that $(\mathcal{T},\mathcal{V})$ is a (C,A_1,B)-pair, $\operatorname{im} E \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \ker H$ and $hc^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$. We clearly have $\mathcal{T}^*(E,A_1,C) \subseteq \mathcal{T}$. The subspace $\mathcal{V}$ is (A_1,B)-invariant. Since $hc^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$, the subspace $\mathcal{V}$ is also

(A_2, B)-invariant. Therefore, we have $\mathcal{V} \subseteq \mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B)$. Hence, we can conclude that

$$\mathcal{T}^*(E, A_1, C) \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B).$$

Sufficiency: Let $(\mathcal{T}, \mathcal{V})$ be the (C, A_1, B)-pair given by $\mathcal{T} = \mathcal{T}^*(E, A_1, C)$ and $\mathcal{V} = \mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B)$. Then we have $\operatorname{im} E \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \ker H$. Since $\mathcal{V}$ is both (A_1, B)-invariant and (A_2, B)-invariant, we have $A_i\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$ for $i = 1, 2$. As such, we obtain $hc^{\mathsf{T}}\mathcal{V} = (A_1 - A_2)\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$. It follows from Theorem 2.13 that the closed-loop system (2.18) is disturbance decoupled. ∎

2.4.2 *Mode-independent dynamic feedback*

As a special case, we consider in this section the linear time-invariant mode-independent feedback controller

$$\dot{w}(t) = Kw(t) + Ly(t) \tag{2.22a}$$

$$u(t) = Mw(t) + Ny(t), \tag{2.22b}$$

where $w \in \mathbb{R}^{n_w}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$, and K, L, M and N are of suitable sizes. By interconnecting this controller with system given by (2.9) and (2.16), we obtain the closed-loop system (2.18) with the system matrices $A_{\mathrm{e},1}$ and $A_{\mathrm{e},2}$ now given by

$$A_{\mathrm{e},i} = \begin{bmatrix} A_i + BNC & BM \\ LC & K \end{bmatrix} \quad \text{for } i = 1, 2. \tag{2.23}$$

We can adapt Theorem 2.13 for mode-dependent dynamic controllers to obtain a similar, but more restrictive, result for mode-independent dynamic controllers.

Theorem 2.15 *There exists a mode-independent dynamic controller of the form (2.22) that renders the system given by (2.9) and (2.16) disturbance decoupled if and only if there exist subspaces $\mathcal{T}$ and $\mathcal{V}$ such that $(\mathcal{T}, \mathcal{V})$ is a (C, A_1, B)-pair, $\operatorname{im} E \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \ker H$ and $hc^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{T}$.*

Proof. A proof of the statement follows from the proof of Theorem 2.13 by taking $L_1 = L_2$, $N_1 = N_2$, $n = 0$ and $\ell = 0$. ∎

Note that the condition $hc^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{T}$ in Theorem 2.15 is more restrictive than the condition $hc^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$ that appears in Theorem 2.13.

Just as for the mode-dependent case, we would like to define some minimal $\mathcal{T}^*$ and maximal $\mathcal{V}^*$ such that $(\mathcal{T}^*, \mathcal{V}^*)$ is a (C, A_1, B)-pair that satisfies the conditions of Theorem 2.15 exactly when the system can be rendered disturbance decoupled by means of a mode-dependent dynamic feedback controller. For this reason, we define the set of subspaces

$$T_{\mathrm{mi}}(E,\{A_1, A_2\},C) := \{\mathcal{T} \subseteq \mathbb{R}^{n_x} \mid \operatorname{im} E \subseteq \mathcal{T}, \\ \text{and } \exists G \text{ s.t. } (A_j + GC)\mathcal{T} \subseteq \mathcal{T} \text{ for } j = 1,2\}. \tag{2.24}$$

Similar to the fact that the set V_{mi} (defined in (2.15)) has a maximal element with respect to subspace addition, the set T_{mi} has a minimal element. Let $\mathcal{T}^*_{\mathrm{mi}}(E, \{A_1, A_2\}, C)$ denote the smallest subspace in T_{mi}. In Section 2.5 we present an algorithm to compute $\mathcal{T}^*_{\mathrm{mi}}$.

The existence of a controller of the form (2.22) that renders the closed-loop system disturbance decoupled does not imply that $(\mathcal{T}^*_{\mathrm{mi}}, \mathcal{V}^*_{\mathrm{mi}})$ is a (C, A_1, B)-pair satisfying the conditions of Theorem 2.15, since $hc^{\mathsf{T}}\mathcal{V}^*_{\mathrm{mi}} \subseteq \mathcal{T}^*_{\mathrm{mi}}$ is not necessarily satisfied. However, the following assertion holds.

Theorem 2.16 *There exists a controller of the form* (2.22) *that renders the system given by* (2.9) *and* (2.16) *disturbance decoupled if and only if at least one of the following two conditions holds*

1. $\mathcal{T}^*_{\mathrm{mi}}(\begin{bmatrix} E & h \end{bmatrix}, \{A_1, A_2\}, C) \subseteq \mathcal{V}^*_{\mathrm{mi}}(H, \{A_1, A_2\}, B)$,
2. $\mathcal{T}^*_{\mathrm{mi}}(E, \{A_1, A_2\}, C) \subseteq \mathcal{V}^*_{\mathrm{mi}}(\begin{bmatrix} H^{\mathsf{T}} & c \end{bmatrix}^{\mathsf{T}}, \{A_1, A_2\}, B)$.

Proof. *Sufficiency:* If the first condition holds, then let

$$\mathcal{T} = \mathcal{T}^*_{\mathrm{mi}}(\begin{bmatrix} E & h \end{bmatrix}, \{A_1, A_2\}, C), \quad \mathcal{V} = \mathcal{V}^*_{\mathrm{mi}}(H, \{A_1, A_2\}, B).$$

Then, we have $h \in \mathcal{T}$ which implies that $hc^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{T}$.

If the second condition holds, let

$$\mathcal{T} = \mathcal{T}^*_{\mathrm{mi}}(E, \{A_1, A_2\}, C), \quad \mathcal{V} = \mathcal{V}^*_{\mathrm{mi}}(\begin{bmatrix} H^{\mathsf{T}} & c \end{bmatrix}^{\mathsf{T}}, \{A_1, A_2\}, B).$$

Then, we have $\mathcal{V} \subseteq \ker c^{\mathsf{T}}$ which implies that $hc^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{T}$.

In both cases we have that $(\mathcal{T}, \mathcal{V})$ is a (C, A_1, B)-pair satisfying $\operatorname{im} E \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \ker H$ and $hc^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{T}$. Therefore, it follows from Theorem 2.15 that there exists a controller of the form (2.22) such that the closed-loop system is disturbance decoupled.

Necessity: Suppose there exists such a controller. By Theorem 2.15, there exist subspaces $\mathcal{T}$ and $\mathcal{V}$ such that $(\mathcal{T}, \mathcal{V})$ is a

(C, A_1, B)-pair, $\operatorname{im} E \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \ker H$, and $hc^\mathsf{T}\mathcal{V} \subseteq \mathcal{T}$. The last condition implies that $hc^\mathsf{T}\mathcal{V} \subseteq \mathcal{V}$ and $hc^\mathsf{T}\mathcal{T} \subseteq \mathcal{T}$, and hence $(\mathcal{T}, \mathcal{V})$ is also a (C, A_2, B)-pair. Therefore, we have $\mathcal{T} \in T_{\mathrm{mi}}(E, \{A_1, A_2\}, \ker H)$ and $\mathcal{V} \in V_{\mathrm{mi}}(H, \{A_1, A_2\}, B)$. Furthermore, $hc^\mathsf{T}\mathcal{V} \subseteq \mathcal{T}$ also implies that we have $h \in \mathcal{T}$ or $\mathcal{V} \subseteq \ker c^\mathsf{T}$. If $h \in \mathcal{T}$, then $\mathcal{T}$ is an element of $T_{\mathrm{mi}}([E \;\; h], \{A_1, A_2\}, C)$, which means that

$$\mathcal{T}^*_{\mathrm{mi}}(\begin{bmatrix} E & h \end{bmatrix}, \{A_1, A_2\}, C) \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \mathcal{V}^*_{\mathrm{mi}}(H, \{A_1, A_2\}, B).$$

If $\mathcal{V} \subseteq \ker c^\mathsf{T}$, then $\mathcal{V} \in V_{\mathrm{mi}}([H^\mathsf{T} \;\; c]^\mathsf{T}, \{A_1, A_2\}, B)$. Hence, we get

$$\mathcal{T}^*_{\mathrm{mi}}(E, \{A_1, A_2\}, C) \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \mathcal{V}^*_{\mathrm{mi}}(\begin{bmatrix} H^\mathsf{T} & c \end{bmatrix}^\mathsf{T}, \{A_1, A_2\}, B).$$

In conclusion, at least one of the two conditions in the statement holds. ∎

2.5 SUBSPACE ALGORITHMS

In this section we first propose subspace algorithms for computing $\mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B)$ and $\mathcal{V}^*_{\mathrm{mi}}(H, \{A_1, A_2\}, B)$. Both algorithms are similar to the invariant subspace algorithm for computing $\mathcal{V}^*(H, A_1, B)$ for linear systems (see e.g. [Trentelman et al., 2001]), and to the subspace algorithms proposed in [Yurtseven et al., 2012] for switched linear systems. Afterwards, we will provide an algorithm for computing $\mathcal{T}^*_{\mathrm{mi}}(E, \{A_1, A_2\}, C)$.

2.5.1 *Algorithm for* $\mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B)$

For computing $\mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B)$, we propose the following algorithm. We first define

$$\mathcal{V}_0 = \ker H. \tag{2.25a}$$

Then, for $i \geqslant 0$, we define

$$\mathcal{V}_{i+1} = \mathcal{V}_i \cap A_1^{-1}(\mathcal{V}_i + \operatorname{im} B), \tag{2.25b}$$

if $h \in \mathcal{V}_i + \operatorname{im} B$, and otherwise

$$\mathcal{V}_{i+1} = \mathcal{V}_i \cap A_1^{-1}(\mathcal{V}_i + \operatorname{im} B) \cap \ker c^\mathsf{T}. \tag{2.25c}$$

It is clear that we have $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$ for all $i \geqslant 0$ and hence there is a $k \leqslant n_x$ such that $\mathcal{V}_k = \mathcal{V}_{k+1}$. Moreover, it follows from the definition of $\mathcal{V}_i$ that we then have $\mathcal{V}_{k+2} = \mathcal{V}_{k+1}$. Therefore, we get $\mathcal{V}_i = \mathcal{V}_k$ for all $i \geqslant k$.

Theorem 2.17 *Let $\mathcal{V}_i$ be defined as in algorithm* (2.25)*. Then for* $q = \min\{k \in \mathbb{N} \mid \mathcal{V}_k = \mathcal{V}_{k+1}\} \leqslant n_x$ *we have*

$$\mathcal{V}_q = \mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B).$$

Proof. As the subspaces $\mathcal{V}_i$ are nested, we have $\mathcal{V}_q \subseteq \mathcal{V}_0 = \ker H$. Since $\mathcal{V}_q$ satisfies $\mathcal{V}_q = \mathcal{V}_{q+1}$, it follows that $\mathcal{V}_q = \mathcal{V}_q \cap A_1^{-1}(\mathcal{V}_q + \operatorname{im} B)$ if $h \in \mathcal{V}_q + \operatorname{im} B$, and $\mathcal{V}_q = \mathcal{V}_q \cap A_1^{-1}(\mathcal{V}_q + \operatorname{im} B) \cap \ker c^\mathsf{T}$ otherwise. In both cases we have $A_1\mathcal{V}_q \subseteq \mathcal{V}_q + \operatorname{im} B$, so $\mathcal{V}_q$ is (A_1, B)-invariant. Furthermore, we have $h \in \mathcal{V}_q + \operatorname{im} B$ or $\mathcal{V}_q \subseteq \ker c^\mathsf{T}$, which implies that $hc^\mathsf{T}\mathcal{V}_q \subseteq \mathcal{V}_q + \operatorname{im} B$. Hence, $A_2\mathcal{V}_q \subseteq A_1\mathcal{V}_q + hc^\mathsf{T}\mathcal{V}_q \subseteq \mathcal{V}_q + \operatorname{im} B$, so $\mathcal{V}_q$ is (A_2, B)-invariant as well. Therefore, we see that $\mathcal{V}_q$ is an element of $V_{\mathrm{md}}(H, \{A_1, A_2\}, B)$, and hence $\mathcal{V}_q \subseteq \mathcal{V}^*_{\mathrm{md}}$.

To prove that we have $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_q$ as well, we use mathematical induction on i. Firstly, we have that $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_0 = \ker H$. Secondly, assume that $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_i$ for some $i \geqslant 0$. Since $\mathcal{V}^*_{\mathrm{md}}$ is both (A_1, B)-invariant and (A_2, B)-invariant, it is (hc^T, B)-invariant as well. Therefore, we have

$$\begin{aligned} hc^\mathsf{T}\mathcal{V}^*_{\mathrm{md}} &\subseteq \mathcal{V}^*_{\mathrm{md}} + \operatorname{im} B \\ &\subseteq \mathcal{V}_i + \operatorname{im} B. \end{aligned}$$

Hence, we get $h \in \mathcal{V}_i + \operatorname{im} B$ or $\mathcal{V}^*_{\mathrm{md}} \subseteq \ker c^\mathsf{T}$. In both cases, it holds that $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_{i+1}$. Therefore, we see that $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_k$ for all $k \geqslant 0$. In particular, we have $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_q$. ∎

2.5.2 *Algorithm for $\mathcal{V}^*_{\mathrm{mi}}(H, \{A_1, A_2\}, B)$*

To compute $\mathcal{V}^*_{\mathrm{mi}}(H, \{A_1, A_2\}, B)$, we refer to Algorithm 5.3 in [Yurtseven et al., 2012], which in our case simplifies to

$$\mathcal{V}_0 = \ker H \tag{2.26a}$$

and

$$\mathcal{V}_{i+1} = \mathcal{V}_i \cap A_1^{-1}(\mathcal{V}_i + \operatorname{im} B) \cap (A_1 - A_2)^{-1}(\mathcal{V}_i) \tag{2.26b}$$

for $i \geqslant 0$.

2.5.3 *Algorithm for $\mathcal{T}^*_{\mathrm{mi}}(E, \{A_1, A_2\}, C)$*

By making use of the well-known duality between controlled invariance and conditioned invariance (see e.g. [Trentelman et al., 2001, Theorem 5.6]), we adapt the algorithm (2.26) for computing $\mathcal{V}^*_{\mathrm{mi}}$ to obtain the following algorithm. We define

$$\mathcal{T}_0 = \operatorname{im} E, \tag{2.27a}$$

and

$$\mathcal{T}_{i+1} = \operatorname{im} E + A_1(\mathcal{T}_i \cap \ker C) + hc^{\mathsf{T}}\mathcal{T}_i \tag{2.27b}$$

for $i \geqslant 0$. It is easy to see that $\mathcal{T}_i \subseteq \mathcal{T}_{i+1}$ for $i \geqslant 0$. Since n_x is finite and $\mathcal{T}_i \in \mathbb{R}^{n_x}$ for all $i \geqslant 0$, it follows that there is a k such that $\mathcal{T}_k = \mathcal{T}_{k+1}$. Furthermore, it follows from the definition of $\mathcal{T}_i$ that we have $\mathcal{T}_i = \mathcal{T}_k$ for all $i \geqslant k$. The next theorem shows that this algorithm indeed gives us $\mathcal{T}^*_{\mathrm{mi}}(E, \{A_1, A_2\}, C)$. We omit the proof, since it follows from similar arguments as employed in the proof of Theorem 2.17.

Theorem 2.18 *Let $\mathcal{T}_i$ be defined as in algorithm (2.27). Then for $q = \min\{k \in \mathbb{N} \mid \mathcal{T}_k = \mathcal{T}_{k+1}\} \leqslant n_x$ we have*

$$\mathcal{T}_q = \mathcal{T}^*_{\mathrm{mi}}(E, \{A_1, A_2\}, C).$$

2.6 CONCLUSIONS

In this chapter, we studied the disturbance decoupling problem for continuous piecewise linear bimodal systems. The main contributions of this chapter include necessary and sufficient conditions for such systems to be disturbance decoupled as well as a complete characterization of the solvability of the disturbance decoupling problem with mode-independent and mode-dependent feedback controllers. Furthermore, we provided subspace algorithms in order to compute the minimal and maximal subspaces that are used in the presented conditions for disturbance decoupling by both state feedback and dynamic feedback.

Future research possibilities include the extension of the presented results to general piecewise affine dynamical systems, which will be the subject of the next chapter.

3

DISTURBANCE DECOUPLING FOR PIECEWISE AFFINE SYSTEMS

ABSTRACT: *In this chapter we study the disturbance decoupling problem for continuous piecewise affine systems. We establish a set of necessary conditions and a set of sufficient conditions, both geometric in nature, for such systems to be disturbance decoupled. Furthermore, we investigate mode-independent state feedback controllers for piecewise affine systems and provide sufficient conditions for the solvability of the disturbance decoupling problem by state feedback. This chapter is based on the conference paper [Everts and Camlibel, 2014a].*

3.1 INTRODUCTION

In this chapter we continue studying the disturbance decoupling problem for linear multi-modal systems. We extend the bimodal systems studied in Chapter 2 to more general continuous piecewise affine systems. Piecewise affine systems are a class of hybrid systems; they are a combination of continuous-time linear systems, the modes, together with the discrete dynamics of switching between these modes.

The switching between the several modes of a piecewise affine system is state-dependent. As discussed in more detail in Sections 1.1, 1.2 and 2.1, this state-dependent switching behavior of piecewise affine systems calls for a different approach than for switched linear systems, for which the switching is state-independent.

In this chapter, we develop a new approach that takes into account the state-dependent switching behavior of piecewise affine systems. This approach allows us to provide a set of necessary conditions and a set of sufficient conditions under which a continuous piecewise affine system is disturbance decoupled. Although these conditions do not coincide in general, we point out some special cases in which they do coincide. Furthermore, we present conditions for the existence of mode-independent static feedback controllers that render the closed-loop system disturbance decoupled. All conditions we present are geometric in nature and easily verifiable.

The following section introduces the class of continuous piecewise affine systems. For this class of systems, we define the disturbance decoupling problem in Section 3.3 and give a set of necessary conditions and a set of sufficient conditions for such a system to be disturbance decoupled. In Section 3.4, we provide conditions under which the necessary conditions and the sufficient conditions coincide. The problem of disturbance decoupling by state feedback is discussed in Section 3.5. Finally, Section 3.6 contains the main conclusions of this chapter.

3.2 PIECEWISE AFFINE SYSTEMS

Before we can define the class of continuous piecewise affine systems, we need the notions of affine functions and piecewise affine functions. An *affine* function is a function $\theta : \mathbb{R}^n \to \mathbb{R}^m$ of the form $\theta(x) = Qx + q$, with Q a $m \times n$ matrix and q an m-vector. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *piecewise affine* if there exists a finite set of affine functions $f_i : \mathbb{R}^n \to \mathbb{R}^m$, $i \in \mathcal{I}_p$, such that for each $x \in \mathbb{R}^n$ we have

$$f(x) \in \{f_1(x), f_2(x), \dots, f_p(x)\}.$$

The domain of a *continuous* piecewise affine function can be divided into a set of polyhedral regions in such a way that the restriction of the function f to any of the regions is given by an affine function [Scholtes, 2012, Prop. 2.2.3]. To make this statement more precise, we quickly review some definitions and results about polyhedral sets.

A *polyhedron* (or polyhedral set) in $\mathbb{R}^n$ is the intersection of a finite number of closed half-spaces. Therefore, a polyhedron P can be represented as $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$, where A is a $p \times n$ matrix and b is a p-vector. The dimension of a polyhedron is equal to the dimension of its affine hull. We call a polyhedron *solid* if it has dimension n. A subset F of a polyhedron P is called a *face* of P if there is a vector $y \in \mathbb{R}^n$ such that

$$F = \{x \in P \mid y^\top x \geqslant y^\top z \text{ for every } z \in P\}.$$

A face of a polyhedron P is also a polyhedron and is called *proper* if its dimension is strictly less than that of P. If a face is $(n-1)$-dimensional we call it a *facet*.

A finite collection $\Xi = \{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_N\}$ of polyhedral sets in $\mathbb{R}^n$ is a *polyhedral subdivision* of $\mathbb{R}^n$ if every polyhedron in Ξ is solid, the union of all polyhedra in Ξ equals $\mathbb{R}^n$, and the intersection of any two polyhedra in Ξ is either empty or a common proper face of both polyhedra (see Figure 3.1).

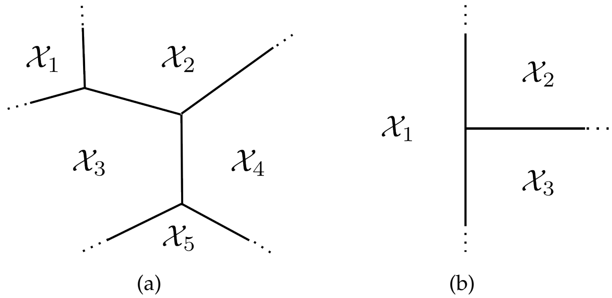

Figure 3.1: (a) Example of a polyhedral subdivision of $\mathbb{R}^2$. (b) This is not a polyhedral subdivision of $\mathbb{R}^2$, since the intersection of $\mathcal{X}_1$ and $\mathcal{X}_2$ is a proper face of $\mathcal{X}_2$ but not of $\mathcal{X}_1$.

As shown in [Scholtes, 2012, Prop. 2.2.3], for a given continuous piecewise affine function f there are a finite number of polyhedral sets $\mathcal{X}_k$ with corresponding matrices $Q_k \in \mathbb{R}^{m\times n}$ and vectors $q_k \in \mathbb{R}^m$ for all $k \in \mathcal{I}_N$, such that $\{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_N\}$ is a polyhedral subdivision of $\mathbb{R}^n$ and f satisfies

$$f(x) = Q_k x + q_k \qquad \forall\, x \in \mathcal{X}_k.$$

A *continuous piecewise affine system* is a system of the form

$$\dot{x}(t) = f(x(t)) + Ed(t) \tag{3.1a}$$
$$z(t) = Hx(t), \tag{3.1b}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $d \in \mathbb{R}^{n_d}$ is the unknown disturbance, $z \in \mathbb{R}^{n_z}$ is the output to be controlled, E and H are matrices of appropriate sizes, and $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is a continuous piecewise affine function.

Since the right-hand side of (3.1a) is Lipschitz continuous in the variable x, for each x_0 and locally integrable disturbance d there exists a unique *absolutely continuous* function $x^{x_0,d}(t)$ satisfying $x(0) = x_0$ and (3.1a) for almost all t. We denote the corresponding output by $z^{x_0,d}(t)$.

As stated above, the function f admits a polyhedral subdivision. So there are polyhedral regions $\mathcal{X}_k$, matrices A_k and vectors g_k, for each $k \in \mathcal{I}_N$, such that $\{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_N\}$ is a polyhedral subdivision of $\mathbb{R}^{n_x}$ and

$$f(x) = A_k x + g_k \qquad \forall x \in \mathcal{X}_k.$$

Hence, we can write system (3.1) as

$$\dot{x}(t) = A_k x(t) + g_k + Ed(t) \qquad \forall x \in \mathcal{X}_k \tag{3.2a}$$
$$z(t) = Hx(t). \tag{3.2b}$$

We can exploit the continuity of $f(x)$ to obtain relations between the matrices A_k. Note that for any two polyhedral regions $\mathcal{X}_k$ and $\mathcal{X}_\ell$ sharing a facet $\mathcal{F}_{k\ell} := \mathcal{X}_k \cap \mathcal{X}_\ell$, we can choose a vector $c_{k\ell}$ and a scalar $f_{k\ell}$ such that the affine hull of $\mathcal{F}_{k\ell}$ is given by the hyperplane

$$\mathcal{H}_{k\ell} := \{x \in \mathbb{R}^{n_x} \mid c_{k\ell}^{\mathsf{T}} x + f_{k\ell} = 0\}.$$

The continuity of f implies that for all $x \in \mathcal{F}_{k\ell} \subseteq \mathcal{H}_{k\ell}$ we have $A_k x + g_k = A_\ell x + g_\ell$, or equivalently

$$(A_k - A_l)x + g_k - g_\ell = 0. \tag{3.3}$$

Since $\mathcal{F}_{k\ell}$ is $(n_x - 1)$-dimensional, it follows that $\ker c_{k\ell}^{\mathsf{T}} \subseteq \ker(A_k - A_\ell)$. Hence, there is a vector $h_{k\ell} \in \mathbb{R}^{n_x}$ such that

$$A_k - A_\ell = h_{k\ell} c_{k\ell}^{\mathsf{T}}. \tag{3.4}$$

By combining this with (3.3) and the fact that $c_{k\ell}^{\mathsf{T}} x + f_{k\ell} = 0$ for all $x \in \mathcal{F}_{k\ell}$, we find that g_k and g_ℓ satisfy

$$g_k - g_\ell = h_{k\ell} f_{k\ell}. \tag{3.5}$$

Notice that, since facet $F_{k\ell}$ is equal to facet $F_{\ell k}$, we can assume that $c_{k\ell} = c_{\ell k}$, $f_{k\ell} = f_{\ell k}$ and $h_{k\ell} = -h_{\ell k}$. If the state x passes from one polyhedral region to another, it will always cross one or more facets, which is why facets will play an important role in this chapter.

Example 3.1 Conewise linear systems are a special class of piecewise linear systems. In such systems, the polyhedral regions are convex cones and the corresponding subsystems are linear. As an example, consider the piecewise linear system with four modes, given by

$$A_1 = \begin{bmatrix} 14 & 11 \\ 10.25 & 8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 6 & 7 \\ 0.25 & 3 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 6 & 0 \\ 0.25 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 14 & 8 \\ 10.25 & 10 \end{bmatrix}, \quad E = 0,$$
$$f_{12} = f_{23} = f_{34} = f_{41} = g_1 = g_2 = g_3 = g_4 = 0,$$
$$c_{23}^{\mathsf{T}} = \begin{bmatrix} 2 & 1 \end{bmatrix}, \; c_{34}^{\mathsf{T}} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; c_{12}^{\mathsf{T}} = c_{41}^{\mathsf{T}} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

See Figure 3.2 for a sketch of the corresponding vector field.

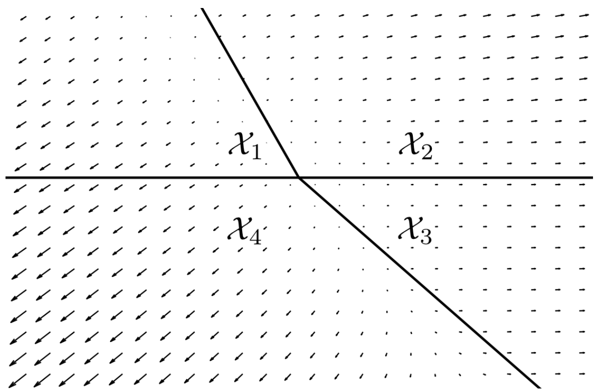

Figure 3.2: Sketch of the vector field corresponding to the continuous piecewise affine system with four modes (without disturbances) in Example 3.1. The line segments denote the facets between the polyhedral regions.

For continuous piecewise affine systems, it is sometimes more convenient to write system (3.2) in the following alternative way:

$$\dot{x}(t) = Ax(t) + Ed(t) + g(y) \tag{3.6a}$$
$$y(t) = Cx(t) \tag{3.6b}$$
$$z(t) = Hx(t), \tag{3.6c}$$

where x, z, E and H are as before, $y \in \mathbb{R}^{n_y}$ is the measured output, A and C are matrices of appropriate sizes, and $g : \mathbb{R}^{n_y} \to \mathbb{R}^{n_x}$ is a continuous piecewise affine function. In this representation, the domain $\mathbb{R}^{n_y}$ of g admits a polyhedral subdivision: there are solid (i.e., n_y-dimensional) polyhedral regions $\mathcal{Y}_k$, matrices $F_k \in \mathbb{R}^{n_x \times n_y}$ and vectors $g_k \in \mathbb{R}^{n_x}$, for all $k \in \mathcal{I}_N$, such that $\{\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_N\}$ is a polyhedral subdivision of $\mathbb{R}^{n_y}$ and

$$g(y) = F_k y + g_k, \qquad \text{if } y \in \mathcal{Y}_k. \tag{3.6d}$$

If $\mathcal{Y}_k$ and $\mathcal{Y}_\ell$ share a facet $\tilde{F}_{k\ell}$, there is a vector $\tilde{c}_{k\ell} \in \mathbb{R}^{n_y}$ and scalar $\tilde{f}_{k\ell}$ such that

$$\tilde{F}_{k\ell} \subseteq \{y \in \mathbb{R}^{n_y} \mid \tilde{c}_{k\ell}^\mathsf{T} y + \tilde{f}_{k\ell} = 0\}.$$

Since g is continuous, we can employ the same reasoning as above to see that there is a vector $\tilde{h}_{k\ell}$ such that

$$F_k - F_\ell = \tilde{h}_{k\ell}\tilde{c}_{k\ell}^\mathsf{T}, \quad g_k - g_\ell = \tilde{h}_{k\ell}\tilde{f}_{k\ell}.$$

To see the equivalence between the two representations, notice that we can write system (3.2) in the form of system (3.6) by taking $C = I$, $A = A_1$, $\mathcal{Y}_k = \mathcal{X}_k$ and $F_k = A_k - A_1$ for $k \in \mathcal{I}_N$. On the other hand, we can write system (3.6) in the form of system (3.2) by letting $A_k = A + F_k C$ and $\mathcal{X}_k = C^{-1}\mathcal{Y}_k$ for $k \in \mathcal{I}_N$, and using the same g_k, E and H. It can be shown that resulting set $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N\}$ is a polyhedral subdivision of $\mathbb{R}^{n_x}$. The corresponding facets $F_{k\ell}$ are equal to $C^{-1}\tilde{F}_{k\ell}$, with $c_{k\ell}^{\mathsf{T}} = \tilde{c}_{k\ell}^{\mathsf{T}} C$ and $f_{k\ell} = \tilde{f}_{k\ell}$.

Combinations of linear systems and static (piecewise linear) nonlinearities, such as saturation, dead-zone and backlash, lead naturally to piecewise affine systems. A concrete example of a continuous piecewise affine system is given next.

Example 3.2 ([Thuan and Camlibel, 2014, Example 2.2]) In high-accuracy motion control of a DC servo system, one has to deal with deadzone-type nonlinear relations between the motor torque T and the current i through the motor windings (see e.g. [Zhonghua et al., 2006]). This can be modeled by the continuous piecewise affine function

$$g(y) = \begin{cases} k_T y - T_- - T_\ell & \text{if } k_T y \leqslant T_- \\ -T_\ell & \text{if } T_- \leqslant k_T y \leqslant T_+ \\ k_T y - T_+ - T_\ell & \text{if } T_+ \leqslant k_T y, \end{cases} \tag{3.7}$$

with k_T the torque constant, T_ℓ the torque applied to the rotor, and T_- and T_+ constant values. If we assume that T_ℓ is constant, we can describe the dynamics of the current i through the motor windings and the angular position θ of the rotor with the following piecewise affine system:

$$\frac{d}{dt}\begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{k_b}{L} \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{B_f}{J} \end{bmatrix}\begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g(y) \end{bmatrix} \tag{3.8a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix}, \tag{3.8b}$$

where $\omega = \dot{\theta}$, J is the moment of inertia of the rotor, and L, R, k_b and B_f are some constants (see [Thuan and Camlibel, 2014] for details).

3.3 THE DISTURBANCE DECOUPLING PROBLEM

We say that a piecewise affine system, given by (3.1), (3.2) or (3.6), is *disturbance decoupled* if for all initial states $x_0 \in \mathbb{R}^{n_x}$ and all locally integrable disturbances d_1 and d_2 we have

$$z^{x_0,d_1}(t) = z^{x_0,d_2}(t), \qquad \forall t \geqslant 0.$$

In this section, we give a necessary condition for a piecewise affine system to be disturbance decoupled, as well as a sufficient condition. Both conditions are geometric in nature.

Theorem 3.3 *If the system (3.2) is disturbance decoupled, then*

$$\sum_{k=1}^{N} \langle A_k \mid \operatorname{im} E \rangle \subseteq \ker H. \tag{3.9}$$

Proof. Let $k \in \mathcal{I}_N$, and let $d_1(t) = d \in \mathbb{R}^{n_d}$ and $d_2 = 0$ be two distinct constant disturbances. Since $\mathcal{X}_k$ is solid, we can choose an interior point x_0 of $\mathcal{X}_k$. Let $x_i(t)$ denote the trajectory $x^{x_0,d_i}(t)$ and let $z_i(t)$ denote the corresponding output, for $i = 1, 2$. Since x_1 and x_2 are continuous, there exists an $\varepsilon > 0$ such that $x_1(t)$ and $x_2(t)$ stay in $\mathcal{X}_k$ for $t \in [0, \varepsilon)$. Thus, the trajectories x_1 and x_2 satisfy

$$\dot{x}_i(t) = A_k x_i(t) + g_k + E d_i(t), \quad \text{for } t \in [0, \varepsilon),\ i = 1, 2.$$

As the system is disturbance decoupled, we have that $z_1(t) = z_2(t)$ and hence

$$H x_1(t) = H x_2(t) \tag{3.10}$$

for all $t \geqslant 0$. Since d_1 and d_2 are constant, we can differentiate equation (3.10) $p \geqslant 1$ times and obtain

$$H A_k^p x_1(t) + H A_k^{p-1} E d = H A_k^p x_2(t)$$

for all $t \in [0, \varepsilon)$. Using $t = 0$ and $x_1(0) = x_2(0)$ we get

$$H A_k^p E d = 0 \qquad \forall p \geqslant 0.$$

Since this holds for any vector $d \in \mathbb{R}^{n_d}$, we conclude that $H A_k^p E = 0$ for all $p \geqslant 0$, and hence by (1.2) we have $\langle A_k \mid \operatorname{im} E \rangle \subseteq \ker H$. By letting k vary over $\{1, \ldots, N\}$, we see that (3.9) holds. ■

For the alternative representation of the system, given by (3.6), we have the following corollary.

Corollary 3.4 *If the system (3.6) is disturbance decoupled, then*

$$\sum_{k=1}^{N} \langle A + F_k C \mid \operatorname{im} E \rangle \subseteq \ker H. \tag{3.11}$$

In general, the subspace $\sum_{k=1}^{N} \langle A_k \mid \operatorname{im} E \rangle$ that appears in Theorem 3.3 is not necessarily invariant under A_k for all $k \in \mathcal{I}_N$. The following theorem shows that such joint invariance relations lead to a sufficient condition.

Theorem 3.5 *The system (3.2) is disturbance decoupled if there is a subspace $\mathcal{V}$ of* $\ker H$ *that contains* $\operatorname{im} E$ *and that is invariant under A_k for all $k \in \mathcal{I}_N$.*

Proof. Let r be the dimension of $\mathcal{V}$ and write $x = \operatorname{col}(v, w)$, where v consists of the first r entries of x. Note that a piecewise affine function is still piecewise affine after a basis transformation. Moreover, the property of disturbance decoupling is invariant under basis transformations as well. Therefore, we can assume without loss of generality that the vectors $\operatorname{col}(v, 0)$ correspond to the subspace $\mathcal{V}$.

Since $\mathcal{V}$ is invariant under each A_k and satisfies $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$, the system matrices are of the form

$$E = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & H_2 \end{bmatrix},$$
$$A_k = \begin{bmatrix} A_{11}^k & A_{12}^k \\ 0 & A_{22}^k \end{bmatrix}, \qquad g_k = \begin{bmatrix} g_1^k \\ g_2^k \end{bmatrix}, \qquad k \in \mathcal{I}_N,$$

where $A_{11}^k \in \mathbb{R}^{r \times r}$, $g_1^k \in \mathbb{R}^r$, $E_1 \in \mathbb{R}^{r \times n_d}$ and $H_2 \in \mathbb{R}^{n_z \times (n_x - r)}$. Hence, we can write system (3.2) as

$$\begin{aligned} \dot{v} &= A_{11}^k v + A_{12}^k w + E_1 d + g_1^k && \forall \operatorname{col}(v, w) \in \mathcal{X}_k \\ \dot{w} &= A_{22}^k w + g_2^k && \forall \operatorname{col}(v, w) \in \mathcal{X}_k \\ z &= H_2 w. \end{aligned}$$

Notice that the output z depends only on w and that w does not directly depend on the disturbance d. However, the disturbance might influence the switching behavior of x and in this way the disturbance might still influence the evolution of w. In the rest of the proof, we will show that this is not the case.

Since the subspace $\mathcal{V}$ is invariant under all A_k, we also have

$$(A_i - A_j)\mathcal{V} \subseteq \mathcal{V}$$

for each i and j. In particular, when the polyhedral regions $\mathcal{X}_i$ and $\mathcal{X}_j$ share a facet F_{ij} we can use equation (3.4) to find that

$$h_{ij}c_{ij}^{\mathsf{T}}\mathcal{V} \subseteq \mathcal{V}.$$

It follows that we have $h_{ij} \in \mathcal{V}$ or $\mathcal{V} \subseteq \ker c_{ij}^{\mathsf{T}}$. We write $h_{ij} = \mathrm{col}(h_{ij,1}, h_{ij,2})$ and $c_{ij} = \mathrm{col}(c_{ij,1}, c_{ij,2})$, where $h_{ij,1}$ and $c_{ij,1}$ are r-dimensional. Note that if $h_{ij} \in \mathcal{V}$, then $h_{ij,2} = 0$. Consequently, using equations (3.4) and (3.5), we see that in this case $A_{22}^i = A_{22}^j$ and $g_2^i = g_2^j$. Next, we will use this observation to define clusters of modes.

We partition $\mathcal{I}_N$ into equivalence classes as follows: i and j are in the same equivalence class if $A_{22}^i = A_{22}^j$ and $g_2^i = g_2^j$. Let $I_1, I_2, \ldots, I_p$ denote the resulting equivalence classes and define clusters $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p$ as

$$\mathcal{C}_\ell = \cup_{k \in I_\ell} \mathcal{X}_k$$

for all $\ell \in \mathcal{I}_p$. Although the union of the clusters is equal to $\mathbb{R}^{n_x}$, $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$ is not necessarily a polyhedral subdivision of $\mathbb{R}^{n_x}$, since a cluster is not necessarily convex. However, within the cluster $\mathcal{C}_\ell$, w satisfies

$$\dot{w} = A_{22}^\ell w + g_2^\ell \quad \forall x \in \mathcal{C}_\ell.$$

Note that in the case that there is just one distinct cluster, equal to $\mathbb{R}^{n_x}$, w satisfies an autonomous affine system, which implies that the system (3.2) is disturbance decoupled.

If there are two or more clusters, then we have $A_{22}^i \neq A_{22}^j$ or $g_2^i \neq g_2^j$ for any facet F_{ij} for which i and j are not in the same equivalence class. Both inequalities imply that $h_{ij,2} \neq 0$, which means that h_{ij} is not an element of $\mathcal{V}$. Consequently, the normal c_{ij} of the facet F_{ij} satisfies $\mathcal{V} \subseteq \ker c_{ij}^{\mathsf{T}}$, and hence c_{ij} must be of the form $c_{ij} = \mathrm{col}(0, c_{ij,2})$. Since there are at least two clusters, there is at least one such cluster-separating facet.

Suppose that a point a is in cluster C_ℓ, but another point b is not. Then the line segment between a and b must intersect a cluster-separating facet F_{ij} for some i and j. For this facet, $c_{ij}^{\mathsf{T}}a + f_{ij}$ and $c_{ij}^{\mathsf{T}}b + f_{ij}$ have different signs. Hence, although the clusters might not be convex, we can determine for each $x \in \mathbb{R}^{n_x}$ in which cluster it resides by only checking the values of $c_{ij}^{\mathsf{T}}x + f_{ij}$ for each cluster-separating facet F_{ij}. For such cluster-separating facets, we have $c_{ij}^{\mathsf{T}}x + f_{ij} = c_{ij,2}^{\mathsf{T}}w + f_{ij}$. Consequently, the value of w is enough to completely determine the cluster that x is in, so w determines the switching between modes. Hence, we have

$$\dot{w} = A_{22}^{\ell} w + g_2^{\ell} \qquad \text{for } \begin{bmatrix} 0 \\ w \end{bmatrix} \in \mathcal{C}_{\ell}. \tag{3.12}$$

Thus we see that w satisfies an autonomous piecewise affine differential equation.

We are now in a position to prove that system (3.1) is disturbance decoupled. Let x_0 be any initial condition and let d_1 and d_2 be two locally integrable disturbances. Denote the two corresponding trajectories by $x_i(t) = x^{x_0,d_i}(t)$, $i = 1,2$, and write $x_i = \operatorname{col}(v_i, w_i)$. From (3.12) we see that $w_1(t) = w_2(t)$ for all $t \geqslant 0$. Consequently, we have $z_1(t) = z_2(t)$ for all $t \geqslant 0$, and hence the system is disturbance decoupled. ∎

For the alternative representation of the system, given by (3.6), we have the following corollary.

Corollary 3.6 *The system (3.6) is disturbance decoupled if there is a subspace $\mathcal{V} \subseteq \ker H$ that contains $\operatorname{im} E$ and that is invariant under $A + F_k C$ for all $k \in \mathcal{I}_N$.*

3.4 NECESSARY AND SUFFICIENT CONDITIONS

The sufficient conditions for system (3.2) to be disturbance decoupled, as given by Theorem 3.5, do not coincide in general with the necessary conditions provided by Theorem 3.3, because $\sum_{i=1}^{N} \langle A_i \mid \operatorname{im} E \rangle$ is not necessarily invariant under each A_i. In this section, we identify a number of particular cases for which the conditions do coincide.

Corollary 3.7 *If $\sum_{i=1}^{N} \langle A_i \mid \operatorname{im} E \rangle$ is invariant under A_i for all $i \in \mathcal{I}_N$, then system (3.2) is disturbance decoupled if and only if*

$$\sum_{i=1}^{N} \langle A_i \mid \operatorname{im} E \rangle \subseteq \ker H.$$

Proof. Theorem 3.3 implies the necessity of the condition. For the sufficiency, let $\mathcal{V} = \sum_{i=1}^{N} \langle A_i \mid \operatorname{im} E \rangle$. Then $\operatorname{im} E \subseteq \mathcal{V}$, and by assumption we have $\mathcal{V} \subseteq \ker H$ and $A_i \mathcal{V} \subseteq \mathcal{V}$ for all $i \in \mathcal{I}_N$. Hence, using Theorem 3.5, we see that system (3.2) is disturbance decoupled. ∎

To investigate when $\sum_{i=1}^{N} \langle A_i \mid \operatorname{im} E \rangle$ is invariant under A_1, $A_2, \ldots, A_N$, we first look at the case that $N = 2$, which corresponds to a bimodal linear system, as discussed in Chapter 2.

Lemma 3.8 *For two square matrices A_1 and A_2 satisfying $A_1 - A_2 = hc^\top$, the subspace $\langle A_1 \mid \operatorname{im} E \rangle + \langle A_2 \mid \operatorname{im} E \rangle$ is invariant under both A_1 and A_2. Furthermore, we have $h \in \langle A_1 \mid \operatorname{im} E \rangle + \langle A_2 \mid \operatorname{im} E \rangle$, or $\langle A_1 \mid \operatorname{im} E \rangle + \langle A_2 \mid \operatorname{im} E \rangle \subseteq \ker c^\top$.*

Proof. Let $\mathcal{V} = \langle A_1 \mid \operatorname{im} E \rangle + \langle A_2 \mid \operatorname{im} E \rangle$. Since $\operatorname{im} A_1 E$ and $\operatorname{im} A_2 E$ are both in $\mathcal{V}$, we see that

$$\operatorname{im} hc^\top E \subseteq \mathcal{V}.$$

This implies that either $h \in \mathcal{V}$, or $c^\top E = 0$. Suppose that $h \notin \mathcal{V}$, then we must have $c^\top E = 0$, which gives us $A_1 E = (A_2 + hc^\top)E = A_2 E$. Since $\operatorname{im} A_1^2 E$ and $\operatorname{im} A_2^2 E$ are both contained in $\mathcal{V}$, we see that $\operatorname{im}(A_1^2 - A_2^2)E \subseteq \mathcal{V}$, so

$$\operatorname{im} hc^\top A_2 E = \operatorname{im}(A_1 - A_2)A_2 E = \operatorname{im}(A_1^2 - A_2^2)E \subseteq \mathcal{V}.$$

Hence, since $h \notin \mathcal{V}$, we have $c^\top A_2 E = 0$, which implies

$$A_1^2 E = A_1 A_2 E = (A_2 + hc^\top)A_2 E = A_2^2 E.$$

By continuing this argument, we see that $c^\top A_2^k E = c^\top A_1^k E = 0$ and $A_2^k E = A_1^k E$ for all $k \geqslant 0$. This implies that $\mathcal{V} = \langle A_1 \mid \operatorname{im} E \rangle = \langle A_2 \mid \operatorname{im} E \rangle$ and $\mathcal{V} \subseteq \ker c^\top$. Hence, we have $h \in \mathcal{V}$ or $\mathcal{V} \subseteq \ker c^\top$. This means that $hc^\top v \in \mathcal{V}$ for any $v \in \mathcal{V}$. Since any $v \in \mathcal{V}$ can be written as $v = v_1 + v_2$, with $v_i \in \langle A_i \mid \operatorname{im} E \rangle$, $i = 1, 2$, we have $A_1 v = A_1 v_1 + A_1 v_2 = A_1 v_1 + A_2 v_2 + hc^\top v_2 \in \mathcal{V}$ for all $v \in \mathcal{V}$. Similarly, we get $A_2 v \in \mathcal{V}$ for all $v \in \mathcal{V}$. Hence, $\mathcal{V}$ is invariant under both A_1 and A_2. ■

Next, we find a sufficient condition for $\sum_{i=1}^{N} \langle A_i \mid \operatorname{im} E \rangle$ to be invariant under all A_i.

Lemma 3.9 *Consider system (3.2). If $h_{k\ell} \in \sum_{i=1}^{N} \langle A_i \mid \operatorname{im} E \rangle$ for all facets $F_{k\ell}$, then $\sum_{i=1}^{N} \langle A_i \mid \operatorname{im} E \rangle$ is invariant under A_i for all $i \in \mathcal{I}_N$.*

Proof. From Theorem 2 in [Shen, 2014] we know that for every $i, j \in \mathcal{I}_N$ there is a finite sequence of indices $k_1, k_2, \ldots, k_{r+1}$

such that $k_1 = i$, $k_{r+1} = j$ and such that $\mathcal{X}_{k_s}$ and $\mathcal{X}_{k_{s+1}}$ share a facet for each $s \in \mathcal{I}_r$. Hence, we can write A_i as

$$A_i = A_j + \sum_{s=1}^{r} h_{k_s k_{s+1}} c_{k_s k_{s+1}}^\top.$$

Therefore, for any element $v_j \in \langle A_j \mid \operatorname{im} E \rangle$ we have

$$A_i v_j = A_j v_j + \sum_{s=1}^{r} h_{k_s k_{s+1}} c_{k_s k_{s+1}}^\top v_j \in \mathcal{V},$$

since $h_{k_s k_{s+1}} \in \mathcal{V}$ for all $s \in \mathcal{I}_r$. Hence, we have $A_i \langle A_j \mid \operatorname{im} E \rangle \subseteq \mathcal{V}$ for every i and j and we can conclude that $\mathcal{V}$ is invariant under each A_i for all $i \in \mathcal{I}_N$. ∎

We now investigate two special cases of systems for which the necessary conditions and sufficient conditions for disturbance decoupling coincide.

Corollary 3.10 *Consider system (3.6). If $C(sI - A)^{-1}E$ is right-invertible as a rational matrix, then system (3.6) is disturbance decoupled if and only if*

$$\sum_{k=1}^{N} \langle A + F_k C \mid \operatorname{im} E \rangle \subseteq \ker H.$$

Proof. We begin by proving the following claim: if $C(sI - A)^{-1}E$ is right-invertible, then so is $C(sI - A - FC)^{-1}E$ for any matrix $F \in \mathbb{R}^{n_y \times n_x}$. For this we use the well-known property

$$(sI - B)^{-1} - (sI - A)^{-1} = (sI - B)^{-1}(B - A)(sI - A)^{-1}.$$

We take $B = A + FC$ and multiply both sides with C from the left and with E from the right. Rearranging the terms then gives us

$$C(sI - A - FC)^{-1}E = \\ \Big(I + C(sI - A - FC)^{-1}F\Big)\, C(sI - A)^{-1}E.$$

Since $I + C(sI - A - FC)^{-1}F$ and $C(sI - A)^{-1}E$ are both right-invertible as a rational matrices, the claim follows.

Let $\mathcal{V} = \sum_{k=1}^{N} \langle A + F_k C \mid \operatorname{im} E \rangle$. From the claim above it follows that $C(sI - A - F_i C)^{-1}E \neq 0$ for each i, so for any facet $\tilde{F}_{k\ell}$ we have $\tilde{c}_{k\ell}^\top C(sI - A - F_i C)^{-1}E \neq 0$ since $\tilde{c}_{k\ell}^\top \neq 0$. Equivalently, $\tilde{c}_{k\ell}^\top C \langle A + F_i C \mid \operatorname{im} E \rangle \neq \{0\}$. Hence, by Lemma

3.8, we see that $h_{k\ell} \in \mathcal{V}$ for all facets $\tilde{F}_{k\ell}$. Then, by Lemma 3.9, $\mathcal{V}$ is invariant under all $A + F_k C$. From Corollaries 3.6 and 3.7 we see that system (3.6) is disturbance decoupled if and only if $\mathcal{V} \subseteq \ker H$. ■

Corollary 3.11 *Consider system (3.2). If all normals c_{ij} to facets F_{ij} are parallel, then system (3.2) is disturbance decoupled if and only if*

$$\sum_{i=1}^{N} \langle A_i \mid \operatorname{im} E \rangle \subseteq \ker H.$$

Proof. Let $\mathcal{V} = \sum_{i=1}^{N} \langle A_i \mid \operatorname{im} E \rangle$. If all normals c_{ij}^{T} to facets F_{ij} are parallel, then all the facets are parallel. This means that the state space is sliced up into parallel regions, each of which shares a facet with at most two other regions.

Suppose that there is a facet F_{ij} for which we have

$$c_{ij}^{\mathsf{T}} \langle A_i \mid \operatorname{im} E \rangle = \{0\},$$

then this implies that $c_{ij}^{\mathsf{T}} A_i^p E = 0$ for all $p \geqslant 0$. It follows that $A_i^p E = A_j^p E$ for all $p \geqslant 0$, which we will prove by mathematical induction. Clearly it holds for $p = 0$. Suppose that it holds for some value of p, then for $p + 1$ we have

$$A_j^{p+1} E = A_j A_j^p E = (A_i - h_{ij} c_{ij}^{\mathsf{T}}) A_i^p E = A_i^{p+1} E,$$

which proves the claim. Consequently, we have that $\langle A_i \mid \operatorname{im} E \rangle = \langle A_j \mid \operatorname{im} E \rangle$, so we also have that

$$c_{ij}^{\mathsf{T}} \langle A_j \mid \operatorname{im} E \rangle = \{0\}.$$

Moreover, if $\mathcal{X}_j$ shares a facet F_{jk} with some other region $\mathcal{X}_k$ as well, then we see that

$$c_{jk}^{\mathsf{T}} \langle A_j \mid \operatorname{im} E \rangle = \{0\},$$

since c_{jk} is a multiple of c_{ij}. By the same reasoning as above, we see that $\langle A_k \mid \operatorname{im} E \rangle = \langle A_j \mid \operatorname{im} E \rangle$. By continuing this argument from region to region, we find that for all facets F_{ij} we have $c_{ij}^{\mathsf{T}} \langle A_i \mid \operatorname{im} E \rangle = \{0\}$ and $\langle A_j \mid \operatorname{im} E \rangle = \langle A_i \mid \operatorname{im} E \rangle$. Hence, we conclude that $\mathcal{V} = \langle A_i \mid \operatorname{im} E \rangle$ for any $i \in \mathcal{I}_N$. As a consequence, $\mathcal{V}$ is invariant under A_i for all $i \in \mathcal{I}_N$.

On the other hand, suppose that $c_{ij}^{\mathsf{T}} \langle A_i \mid \operatorname{im} E \rangle \neq \{0\}$ for all facets F_{ij}. From Lemma 3.8 we know that $h_{ij} \in \mathcal{V}$ for all facets.

Using Lemma 3.9, we see that also in this case $\mathcal{V}$ is invariant under A_i for all $i \in \mathcal{I}_N$.

Hence, in both cases $\mathcal{V}$ is invariant under all A_i. By Corollary 3.7, system (3.6) is disturbance decoupled if and only if $\mathcal{V} \subseteq \ker H$. ■

Example 3.12 We consider the system as given in Example 3.2 and add a disturbance d:

$$\frac{d}{dt}\begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{k_b}{L} \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{B_f}{J} \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g(y) \end{bmatrix} + Ed \tag{3.13a}$$

$$y = z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i & \theta & \omega \end{bmatrix}^{\mathsf{T}}, \tag{3.13b}$$

with $E \in \mathbb{R}^{n_x \times 1}$ and $g(y)$ as in (3.7). To illustrate the theory developed in this section, we discuss whether this system is disturbance decoupled for three choices for E: E_1, E_2 and E_3, where E_i is the ith column of the 3×3 identity matrix. First note that the system satisfies the conditions in Corollary 3.11. Hence, we only have to check if $\sum_{k=1}^{3} \langle A_k \mid \operatorname{im} E \rangle \subseteq \ker H$, where $A_k = A + F_k C$. For E_2, we have $\langle A_k \mid \operatorname{im} E \rangle = \operatorname{im} E \subseteq \ker H$ for $k = 1,2,3$. Therefore, $\sum_{k=1}^{3} \langle A_k \mid \operatorname{im} E \rangle$ equals $\ker H$, implying that the system is disturbance decoupled. For E_1, we see that $\operatorname{im} E_1 \not\subseteq \ker H$, hence $\sum_{k=1}^{3} \langle A_k \mid \operatorname{im} E \rangle \not\subseteq \ker H$. For E_3, we have $\operatorname{im} E \subseteq \ker H$, but $\sum_{k=1}^{3} \langle A_k \mid \operatorname{im} E \rangle = \mathbb{R}^{n_x} \not\subseteq \ker H$. Consequently, the system is not disturbance decoupled for both $E = E_1$ and $E = E_3$.

3.5 STATE FEEDBACK

In this section, we discuss the problem of finding a state feedback law that renders a given piecewise affine system disturbance decoupled. We consider the continuous piecewise affine system

$$\dot{x}(t) = f(x(t)) + Ed(t) + Bu(t) \tag{3.14a}$$

$$z(t) = Hx(t), \tag{3.14b}$$

with x, d, z, E and H as before, $u \in \mathbb{R}^{n_u}$ the input, $B \in \mathbb{R}^{n_x \times n_u}$, and $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ a piecewise affine function. Like before, the function f admits a polyhedral subdivision $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N\}$ of $\mathbb{R}^{n_x}$. For each region $\mathcal{X}_k$, there are matrices A_k and vectors

g_k such that $f(x) = A_k x + g_k$ for all $x \in \mathcal{X}_k$. Using this, we can write system (3.14) as

$$\dot{x}(t) = A_k x(t) + g_k + Ed(t) + Bu(t) \qquad \forall x \in \mathcal{X}_k \quad (3.15a)$$
$$z(t) = Hx(t). \quad (3.15b)$$

In the rest of this section we will investigate conditions for the existence of a mode-independent state feedback law that renders system (3.15) disturbance decoupled.

We consider a mode-independent feedback law $u = Kx$, for some matrix $K \in \mathbb{R}^{n_u \times n_x}$. Applying such a feedback law to system (3.15) results in the following closed-loop system

$$\dot{x}(t) = (A_k + BK)x(t) + g_k + Ed(t) \qquad \forall x \in \mathcal{X}_k \quad (3.16a)$$
$$z(t) = Hx(t). \quad (3.16b)$$

In view of Theorem 3.5 we see that system (3.16) is disturbance decoupled if there is a subspace $\mathcal{V}$ that satisfies

$$(A_i + BK)\mathcal{V} \subseteq \mathcal{V}, \qquad \forall i \in \mathcal{I}_N \quad (3.17)$$
$$\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H. \quad (3.18)$$

To investigate whether such a subspace with a corresponding matrix K exists, we define the following set of subspaces:

$$V(H, \{A_k\}_{k=1}^N, B) = \{\mathcal{V} \subseteq \ker H \mid \exists K \in \mathbb{R}^{n_u \times n_x} \text{ s.t. } (A_k + BK)\mathcal{V} \subseteq \mathcal{V} \text{ for all } k \in \mathcal{I}_N\}.$$

It is easy to see that $\mathcal{V} \in V(H, \{A_k\}_{k=1}^N, B)$ if and only if $A_1\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$ and $(A_i - A_j)\mathcal{V} \subseteq \mathcal{V}$ for all $i, j \in \mathcal{I}_N$. It follows that the set $V(H, \{A_k\}_{k=1}^N, B)$ is closed under subspace addition. Thus we can define $\mathcal{V}^*(H, \{A_k\}_{k=1}^N, B)$ to be the largest element in $V(H, \{A_k\}_{k=1}^N, B)$. We observe that $\operatorname{im} E \subseteq \mathcal{V}^*(H, \{A_k\}_{k=1}^N, B)$ if and only if there is a subspace $\mathcal{V}$ satisfying (3.17)-(3.18). Hence, we arrive at the following theorem.

Theorem 3.13 *There exists a feedback law $u = Kx$ that renders the system (3.16) disturbance decoupled if*

$$\operatorname{im} E \subseteq \mathcal{V}^*(H, \{A_k\}_{k=1}^N, B).$$

Remark 3.14 To compute $\mathcal{V}^*(H, \{A_k\}_{k=1}^N, B)$, we refer to [Yurtseven et al., 2012, Algorithm 5.3].

Example 3.15 We extend on Example 1 and 2, by adding state feedback in the form of applying a voltage v to the motor. This results in the system

$$\frac{d}{dt}\begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{k_b}{L} \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{B_f}{J} \end{bmatrix}\begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g(y) \end{bmatrix} + Ed + Bu \tag{3.19a}$$

$$y = z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} i & \theta & \omega \end{bmatrix}^{\mathsf{T}}, \tag{3.19b}$$

where we choose $B = [1\ 0\ 0]^{\mathsf{T}}$ and $E = [0\ 0\ 1]^{\mathsf{T}}$. Then for $K = [0\ 0\ k_b/L]$, we have that $(A_i + BK)\ker H \subseteq \ker H$ for $i = 1, 2, 3$, hence $\ker H \subseteq \mathcal{V}^*(H, \{A_1, A_2, A_3\}, B)$. On the other hand, $\mathcal{V}^*(H, \{A_1, A_2, A_3\}, B)$ is contained in $\ker H$. Therefore, we have $\mathcal{V}^*(H, \{A_1, A_2, A_3\}, B) = \ker H$. Since $\operatorname{im} E \subseteq \ker H$, we conclude that the feedback $u = Kx$ renders the system disturbance decoupled.

3.6 CONCLUSIONS

In this paper, we established necessary conditions as well as sufficient conditions for a continuous piecewise affine system to be disturbance decoupled. These conditions do not coincide in general. However, we identified a number of particular cases for which they do coincide. Furthermore, we provided sufficient conditions for the existence of a mode-independent static feedback controller that renders a given piecewise affine system disturbance decoupled. All presented conditions are geometric in nature and can be easily verified by utilizing extensions of the well-known subspace algorithms.

Further research possibilities include investigating the gap between the necessary conditions and sufficient conditions as well as studying mode-dependent state feedback for disturbance decoupling.

In the next chapter we will study the disturbance decoupling problem for a particular class of linear complementarity problems, which are closely related to the piecewise affine systems in this chapter.

4 DISTURBANCE DECOUPLED LINEAR COMPLEMENTARITY SYSTEMS

ABSTRACT: *In this chapter we study the disturbance decoupling problem for a particular class of linear complementarity systems. We rewrite the linear complementarity system as a linear multi-modal system and provide crisp necessary and sufficient conditions for such a system to be disturbance decoupled. This chapter is based on the book chapter [Everts and Camlibel, 2015], dedicated to Prof. Dr. Harry L. Trentelman on the occasion of his sixtieth birthday.*

4.1 INTRODUCTION

In this chapter we study the disturbance decoupling problem for a particular class of linear complementarity systems. Linear complementarity systems are nonsmooth dynamical systems that are obtained by taking a standard linear input/output system and imposing certain complementarity relations on a number of input/output pairs at each time instant. A wealth of examples, from various areas of engineering as well as operations research, of linear complementarity systems can be found in [Camlibel et al., 2004; Schumacher, 2004; van der Schaft and Schumacher, 2000; Heemels and Brogliato, 2003]. For the work on the analysis of linear complementarity systems, we refer to [Camlibel et al., 2003; Heemels et al., 2002; Camlibel et al., 2002; van der Schaft and Schumacher, 1996; Camlibel, 2007; van der Schaft and Schumacher, 1998; Heemels et al., 2000].

Particular linear complementarity systems can be written as linear multi-modal systems, namely those of index zero [Camlibel, 2001, Chapter 2]. Different from the piecewise affine systems treated in Chapter 3, the resulting polyhedral regions on which the modes are active, can now be non-solid, and together they do not cover the full state space. The linear subsystems of a linear complementarity system share a certain geometric structure. By exploiting this geometric structure, we provide a necessary and sufficient condition for disturbance decoupledness that is crisp and easily checkable.

The structure of this chapter is as follows. In Section 4.2 we start with the formulation of the linear complementarity problem, and introduce linear complementarity systems. In

Section 4.3 we first define what we mean by a linear complementarity system to be disturbance decoupled. After providing some technical auxiliary results that are, in a way, of interest themselves, we present a necessary and sufficient condition for disturbance decoupledness, which is the main result of this chapter. Finally, the chapter closes with conclusions in Section 4.4.

4.2 LINEAR COMPLEMENTARITY PROBLEM/SYSTEM

The problem of finding a vector $z \in \mathbb{R}^{n_z}$ such that

$$z \geqslant 0 \tag{4.1a}$$
$$q + Mz \geqslant 0 \tag{4.1b}$$
$$z^\mathsf{T}(q + Mz) = 0 \tag{4.1c}$$

for a given vector $q \in \mathbb{R}^{n_z}$ and a matrix $M \in \mathbb{R}^{n_z \times n_z}$ is known as the *linear complementarity problem*. Here, the inequalities for vectors are componentwise inequalities. We denote (4.1) by $\text{LCP}(q, M)$. It is well-known [Cottle et al., 1992, Thm. 3.3.7] that the $\text{LCP}(q, M)$ admits a unique solution for each q if and only if all principal minors of M are positive. Such matrices are called P-matrices in the literature of the mathematical programming. It is well-known (see for instance [Cottle et al., 1992, Thm. 3.1.6 and Thm. 3.3.7]) that every positive definite matrix is in this class.

When the matrix M is a P-matrix, the unique solution $z(q)$ of the $\text{LCP}(q, M)$ depends on q in a Lipschitz continuous way. In particular, for each q there exists an index set $\alpha \subseteq \mathcal{I}_{n_z}$ such that the solution $z = z(q)$ satisfies

$$\begin{aligned} z_\alpha &\geqslant 0, & (q + Mz)_\alpha &= 0, \\ z_{\alpha^c} &= 0, & (q + Mz)_{\alpha^c} &\geqslant 0, \end{aligned}$$

or equivalently,

$$z_\alpha = -(M_{\alpha\alpha})^{-1} q_\alpha, \qquad -(M_{\alpha\alpha})^{-1} q_\alpha \geqslant 0, \tag{4.2a}$$
$$z_{\alpha^c} = 0, \qquad q_{\alpha^c} - M_{\alpha^c\alpha}(M_{\alpha\alpha})^{-1} q_\alpha \geqslant 0, \tag{4.2b}$$

where α^c denotes the set $\mathcal{I}_{n_z} \setminus \alpha$.

Linear complementarity systems (LCSs) are nonsmooth dynamical systems that are obtained in the following way. Take a standard linear input/output system. Select a number of input/output pairs (z_i, w_i), and impose for each of these pairs a complementarity relation of the type (4.1) at each time instant. In this chapter we will focus on the LCSs of the following form:

$$\dot{x}(t) = Ax(t) + Bz(t) + Ed(t) \tag{4.3a}$$
$$w(t) = Cx(t) + Dz(t) + Fd(t) \tag{4.3b}$$
$$0 \leqslant z(t) \perp w(t) \geqslant 0 \tag{4.3c}$$
$$y(t) = Jx(t). \tag{4.3d}$$

Here $x \in \mathbb{R}^{n_x}$ is the state, $(z, w) \in \mathbb{R}^{2n_z}$ are the complementarity variables, $d \in \mathbb{R}^{n_d}$ is the disturbance, $y \in \mathbb{R}^{n_y}$ is the output, $\perp$ denotes orthogonality and all the matrices are of appropriate sizes.

In the sequel we will work under the following blanket assumptions:

1. The matrix D is a P-matrix.

2. The transfer matrix $F + C(sI - A)^{-1}E$ is right-invertible as a rational matrix.

Since D is a P-matrix, $z(t)$ is a piecewise linear function of $Cx(t) + Fd(t)$ (see e.g. [Cottle et al., 1992]). This means that for each initial state x_0 and locally-integrable disturbance d there exist unique absolutely continuous trajectories $(x^{x_0,d}, y^{x_0,d})$ and locally-integrable trajectories $(z^{x_0,d}, w^{x_0,d})$ such that $x^{x_0,d}(0) = x_0$ and the quadruple $(x^{x_0,d}, z^{x_0,d}, w^{x_0,d}, y^{x_0,d})$ satisfies the relations (4.3) for almost all $t \geqslant 0$.

Although LCSs are nonsmooth and nonlinear, their local linear behavior enables elegant characterizations of certain system-theoretic properties. In the next section we will study the disturbance decoupling problem for LCSs.

4.3 DISTURBANCE DECOUPLED LCSS

We say that an LCS (4.3) is *disturbance decoupled* if for all initial states x_0 and all locally integrable disturbances d_1 and d_2 we have

$$y^{x_0,d_1}(t) = y^{x_0,d_2}(t), \qquad \forall t \geqslant 0.$$

In this section, we will investigate necessary and sufficient conditions for an LCS (4.3) to be disturbance decoupled. To do so, we first derive an alternative representation of an LCS. This representation is closely related to the piecewise affine systems in Chapter 3 and makes the underlying switching behavior more transparent.

Since D is a P-matrix, we can solve the LCP given by (4.3c) by employing (4.2). To simplify notation later on, we first define the following matrices for a given index set $\alpha \subseteq \mathcal{I}_{n_z}$:

$$N_\alpha = -B_{\bullet\alpha}(D_{\alpha\alpha})^{-1}I_{\alpha\bullet} \tag{4.4}$$

$$A_\alpha = A + N_\alpha C \tag{4.5}$$

$$E_\alpha = E + N_\alpha F \tag{4.6}$$

$$G_\alpha = \begin{bmatrix} -(D_{\alpha\alpha})^{-1}C_{\alpha\bullet} \\ C_{\alpha^c\bullet} - D_{\alpha^c\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\bullet} \end{bmatrix} \tag{4.7}$$

$$H_\alpha = \begin{bmatrix} -(D_{\alpha\alpha})^{-1}F_{\alpha\bullet} \\ F_{\alpha^c\bullet} - D_{\alpha^c\alpha}(D_{\alpha\alpha})^{-1}F_{\alpha\bullet} \end{bmatrix}, \tag{4.8}$$

where a subscript $\alpha\beta$ selects rows α and columns β of a matrix, for given index sets α and β. Furthermore, the $\bullet$ means selecting all rows or columns and α^c denotes the complement of α in $\mathcal{I}_{n_z}$.

If the quadruple (x, d, z, w) satisfies (4.3a)-(4.3c) for almost all $t \geqslant 0$ then for almost all $t \geqslant 0$ there exists an index set $\alpha_t \subseteq \mathcal{I}_{n_z}$ such that

$$\dot{x}(t) = A_{\alpha_t}x(t) + E_{\alpha_t}d(t) \text{ when } \begin{bmatrix} G_{\alpha_t} & H_{\alpha_t} \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} \geqslant 0. \tag{4.9}$$

The resulting system (4.9) is a linear multi-modal system. Different from the piecewise affine systems in Chapter 3, the underlying polyhedral regions are not all solid and their union is not equal to $\mathbb{R}^{n_x}$. However, the linear subsystems of (4.9) share a certain geometric structure, which we will exploit to prove the following auxiliary result concerning the subspace $\sum_{\gamma \subseteq \mathcal{I}_{n_z}} \langle A_\gamma \mid \text{im } E_\gamma \rangle$, which plays an important role in the disturbance decoupling problem later on.

Lemma 4.1 *Let $\mathcal{S} = \sum_{\gamma \subseteq \mathcal{I}_{n_z}} \langle A_\gamma \mid \text{im } E_\gamma \rangle$. The following statements hold:*

1. $\text{im } (N_\alpha - N_\beta) \subseteq \mathcal{S}$ *for any* $\alpha, \beta \subseteq \mathcal{I}_{n_z}$.
2. $\mathcal{S}$ *is invariant under* A_α *for any* $\alpha \subseteq \mathcal{I}_{n_z}$.
3. $\mathcal{S} = \langle A \mid \text{im } \begin{bmatrix} B & E \end{bmatrix} \rangle$.

Proof. To prove the first statement, let Σ_γ denote the linear system $\Sigma(A_\gamma, E_\gamma, C, F)$ for $\gamma \subseteq \mathcal{I}_{n_z}$. It follows from (1.7) that

$$\mathcal{T}^*(\Sigma_\gamma) \subseteq \langle A_\gamma \mid \text{im } E_\gamma \rangle \subseteq \mathcal{S}.$$

Then, we have

$$\begin{aligned}(A+N_\gamma C)\mathcal{T}^*(\Sigma_\gamma) &= (A+N_\gamma C)\langle A_\gamma \mid \text{im } E_\gamma\rangle \\ &\subseteq \langle A_\gamma \mid \text{im } E_\gamma\rangle \subseteq \mathcal{S}.\end{aligned}$$

Let $\widetilde{\Sigma}$ denote the linear system $\Sigma(A,E,C,F)$. It follows from (1.6) that $\mathcal{T}^*(\Sigma_\gamma) = \mathcal{T}^*(\widetilde{\Sigma})$ and hence that

$$(A+N_\gamma C)\mathcal{T}^*(\widetilde{\Sigma}) \subseteq \mathcal{S}$$

for any $\gamma \subseteq \mathcal{I}_{n_z}$. This yields

$$(N_\alpha - N_\beta)C\mathcal{T}^*(\widetilde{\Sigma}) \subseteq \mathcal{S} \tag{4.10}$$

for any $\alpha, \beta \subseteq \mathcal{I}_{n_z}$. Also we have

$$\text{im } (E+N_\gamma F) \subseteq \langle A_\gamma \mid \text{im } E_\gamma\rangle \subseteq \mathcal{S}$$

for any $\gamma \subseteq \mathcal{I}_{n_z}$. Thus, we get

$$(N_\alpha - N_\beta)\text{im } F \subseteq \mathcal{S}.$$

By combining the last relation with (4.10), we obtain

$$(N_\alpha - N_\beta)\big(\text{im } F + C\mathcal{T}^*(\widetilde{\Sigma})\big) \subseteq \mathcal{S}.$$

Since the transfer matrix $F + C(sI-A)^{-1}E$ is right-invertible as a rational matrix, it follows from (1.9) that $\text{im } F + C\mathcal{T}^*(\widetilde{\Sigma}) = \mathbb{R}^{n_y}$. Therefore, we have

$$\text{im } (N_\alpha - N_\beta) \subseteq \mathcal{S}.$$

To prove the second statement, let $\alpha, \gamma \subseteq \mathcal{I}_{n_z}$. Note that

$$\begin{aligned}A_\alpha\langle A_\gamma \mid \text{im } E_\gamma\rangle &\subseteq A_\gamma\langle A_\gamma \mid \text{im } E_\gamma\rangle + \text{im } (A_\alpha - A_\gamma) \\ &\subseteq \langle A_\gamma \mid \text{im } E_\gamma\rangle + \text{im } (N_\alpha - N_\gamma).\end{aligned}$$

It follows from the definition of $\mathcal{S}$ and the first statement that

$$A_\alpha\langle A_\gamma \mid \text{im } E_\gamma\rangle \subseteq \mathcal{S}.$$

Hence, we have

$$\begin{aligned}A_\alpha\mathcal{S} &\subseteq A_\alpha\Big(\sum_{\gamma\subseteq\mathcal{I}_{n_z}}\langle A_\gamma \mid \text{im } E_\gamma\rangle\Big) \\ &\subseteq \sum_{\gamma\subseteq\mathcal{I}_{n_z}} A_\alpha\langle A_\gamma \mid \text{im } E_\gamma\rangle \subseteq \mathcal{S}.\end{aligned}$$

To prove the third statement, note first that im $N_\gamma \subseteq$ im B for any $\gamma \subseteq \mathcal{I}_{n_z}$. Hence, we have

$$\text{im } E_\gamma = \text{im } (E + N_\gamma C) \subseteq \text{im } \begin{bmatrix} B & E \end{bmatrix}.$$

This results in

$$\langle A_\gamma \mid \text{im } E_\gamma \rangle \subseteq \langle A_\gamma \mid \text{im } \begin{bmatrix} B & E \end{bmatrix} \rangle \tag{4.11}$$

for any $\gamma \subseteq \mathcal{I}_{n_z}$. Since $A_\gamma = A + N_\gamma C$ and im $N_\gamma \subseteq$ im B, it follows from (1.3) that

$$\langle A_\gamma \mid \text{im } \begin{bmatrix} B & E \end{bmatrix} \rangle = \langle A \mid \text{im } \begin{bmatrix} B & E \end{bmatrix} \rangle.$$

In view of (4.11), this means that

$$\langle A_\gamma \mid \text{im } E_\gamma \rangle \subseteq \langle A \mid \text{im } \begin{bmatrix} B & E \end{bmatrix} \rangle$$

for any $\gamma \subseteq \mathcal{I}_{n_z}$. Consequently, we obtain

$$\mathcal{S} = \sum_{\gamma \subseteq \mathcal{I}_{n_z}} \langle A_\gamma \mid \text{im } E_\gamma \rangle \subseteq \langle A \mid \text{im } \begin{bmatrix} B & E \end{bmatrix} \rangle. \tag{4.12}$$

It follows from the fact that $\langle A_\gamma \mid \text{im } E_\gamma \rangle \subseteq \mathcal{S}$ and that

$$\begin{aligned} &\langle A \mid \text{im } E \rangle \subseteq \mathcal{S} \\ &\langle A - BD^{-1}C \mid \text{im } (E - BD^{-1}F) \rangle \subseteq \mathcal{S} \end{aligned}$$

for the particular choices $\gamma = \varnothing$ and $\gamma = \mathcal{I}_{n_z}$, respectively. We know from (1.6) that the strongly reachable subspaces of the systems $\Sigma(A, E, C, F)$ and $\Sigma(A - BD^{-1}C, E - BD^{-1}F, C, F)$ coincide. Let $\mathcal{T}^*$ denote this common strongly reachable subspace. It follows from (1.7) that

$$\begin{aligned} &\mathcal{T}^* \subseteq \langle A \mid \text{im } E \rangle \subseteq \mathcal{S} \\ &\mathcal{T}^* \subseteq \langle A_{\mathcal{I}_{n_z}} \mid \text{im } E_{\mathcal{I}_{n_z}} \rangle \subseteq \mathcal{S}. \end{aligned}$$

These inclusions yield

$$\begin{aligned} &A\mathcal{T}^* \subseteq A\langle A \mid \text{im } E \rangle \subseteq \langle A \mid \text{im } E \rangle \subseteq \mathcal{S} \\ &A_{\mathcal{I}_{n_z}}\mathcal{T}^* \subseteq A_{\mathcal{I}_{n_z}}\langle \mathcal{I}_{n_z} \mid \text{im } E_{\mathcal{I}_{n_z}} \rangle \subseteq \langle A_{\mathcal{I}_{n_z}} \mid \text{im } E_{\mathcal{I}_{n_z}} \rangle \subseteq \mathcal{S}. \end{aligned}$$

Using $A - A_{\mathcal{I}_{n_z}} = BD^{-1}C$, we can conclude that

$$BD^{-1}C\mathcal{T}^* \subseteq \mathcal{S}. \tag{4.13}$$

On the other hand, we readily have

$$\begin{aligned} &\text{im}\, E \subseteq \langle A \mid \text{im}\, E \rangle \subseteq \mathcal{S} \\ &\text{im}(E - BD^{-1}F) \subseteq \langle A - BD^{-1}C \mid \text{im } (E - BD^{-1}F) \rangle \subseteq \mathcal{S}. \end{aligned}$$

Combining these two inclusions results in

$$BD^{-1}\text{im } F \subseteq \mathcal{S}.$$

Together with (4.13), this implies that

$$BD^{-1}(\text{im } F + C\mathcal{T}^*) \subseteq \mathcal{S}.$$

It follows from the blanket assumption and (1.9) that

$$\text{im } F + C\mathcal{T}^* = \mathbb{R}^{n_y}.$$

Thus, we get

$$\text{im } B \subseteq \mathcal{S}.$$

From the second statement of Lemma 4.1, we know that the subspace $\mathcal{S}$ is A_α-invariant for any $\alpha \subseteq \mathcal{I}_{n_z}$. In particular, the choice of $\alpha = \varnothing$ implies that $\mathcal{S}$ is A-invariant. Since $\langle A \mid \text{im } B\rangle$ is the smallest A-invariant subspace that contains im B, we have

$$\langle A \mid \text{im } B\rangle \subseteq \mathcal{S}. \tag{4.14}$$

As we readily have

$$\langle A \mid \text{im } E\rangle \subseteq \mathcal{S},$$

the inclusion (4.14) implies that

$$\langle A \mid \text{im } B\rangle + \langle A \mid \text{im } E\rangle = \langle A \mid \text{im } \begin{bmatrix} B & E \end{bmatrix}\rangle \subseteq \mathcal{S}.$$

Together with (4.12), this proves that

$$\mathcal{S} = \langle A \mid \text{im } \begin{bmatrix} B & E \end{bmatrix}\rangle.$$ ■

Now we are ready to present necessary and sufficient conditions for an LCS to be disturbance decoupled.

Theorem 4.2 *An LCS of the form* (4.3) *is disturbance decoupled if and only if*

$$\langle A \mid \text{im } \begin{bmatrix} B & E \end{bmatrix}\rangle \subseteq \ker J.$$

Proof. *Necessity:* Let $\gamma \subseteq \mathcal{I}_{n_z}$. Note that

$$\begin{bmatrix} G_\gamma & H_\gamma \end{bmatrix} = \begin{bmatrix} -(D_{\gamma\gamma})^{-1} & 0 \\ -D_{\gamma^c\gamma}(D_{\gamma\gamma})^{-1} & I \end{bmatrix} \begin{bmatrix} C_{\gamma\bullet} & F_{\gamma\bullet} \\ C_{\gamma^c\bullet} & F_{\gamma^c\bullet} \end{bmatrix}. \tag{4.15}$$

Since $F + C(sI - A)^{-1}E$ is right-invertible as a rational matrix by the blanket assumption, $[C \;\; F]$ is of full row rank. So must be the matrix $[G_\gamma \;\; H_\gamma]$ due to (4.15). Then, one can find x_0 and d such that

$$\begin{bmatrix} G_\gamma & H_\gamma \end{bmatrix} \begin{bmatrix} x_0 \\ d \end{bmatrix} > 0.$$

Let $e \in \mathbb{R}^{n_z}$. Clearly, there exists a sufficiently small $\mu > 0$ such that

$$\begin{bmatrix} G_\gamma & H_\gamma \end{bmatrix} \begin{bmatrix} x_0 \\ d + \mu e \end{bmatrix} > 0.$$

Now define

$$d_1(t) = d \quad \text{and} \quad d_2(t) = d + \mu e$$

for all $t \geqslant 0$. Let $x_i(t)$ denote the trajectory $x^{x_0,d_i}(t)$ for $i = 1, 2$. Since x_i and d_i are continuous, there exists an $\epsilon > 0$ such that

$$\begin{bmatrix} G_\gamma & H_\gamma \end{bmatrix} \begin{bmatrix} x_i(t) \\ d_i(t) \end{bmatrix} > 0$$

holds for all $t \in [0, \epsilon)$. Thus, the trajectories x_1 and x_2 satisfy

$$\dot{x}_i(t) = A_\gamma x_i(t) + E_\gamma d_i(t)$$

for all $t \in [0, \epsilon)$ and $i = 1, 2$. As the system is disturbance decoupled, we have that

$$J x_1(t) = J x_2(t)$$

for all $t \in [0, \epsilon)$. Since d_1 and d_2 are constant, we obtain

$$J(A_\gamma x_0 + E_\gamma d) = J\big(A_\gamma x_0 + E_\gamma (d + \mu e)\big)$$

by differentiating and evaluating at $t = 0$. This results in

$$J E_\gamma e = 0.$$

By repeating the differentiation and evaluation at $t = 0$, we get

$$J A_\gamma^k E_\gamma e = 0$$

for all $k \geqslant 0$. Since e is arbitrary, we have

$$J A_\gamma^k E_\gamma = 0$$

for all $k \geqslant 0$. Consequently, one gets

$$\langle A_\gamma \mid \text{im } E_\gamma \rangle \subseteq \ker J.$$

Thus, we have

$$\sum_{\gamma \subseteq \mathcal{I}_{n_z}} \langle A_\gamma \mid \text{im } E_\gamma \rangle \subseteq \ker J.$$

It follows from the third statement of Lemma 4.1 that

$$\langle A \mid \text{im } \begin{bmatrix} B & E \end{bmatrix} \rangle \subseteq \ker J.$$

Sufficiency: It is enough to show that

$$x^{x_0,d_1}(t) - x^{x_0,d_2}(t) \in \langle A \mid \text{im } \begin{bmatrix} B & E \end{bmatrix} \rangle, \qquad \forall t \geqslant 0$$

for any initial state $x_0 \in \mathbb{R}^{n_x}$ and all locally-integrable disturbances d_1 and d_2. To do so, let $\mathcal{V} := \langle A \mid \text{im } \begin{bmatrix} B & E \end{bmatrix} \rangle$ and let $v \in \mathcal{V}^\perp$. From (4.3a), we have

$$v^\top \big(\dot{x}^{x_0,d_1}(t) - \dot{x}^{x_0,d_2}(t)\big) = v^\top A \big(x^{x_0,d_1}(t) - x^{x_0,d_2}(t)\big) \tag{4.16}$$

for almost all $t \geqslant 0$. Define

$$\zeta(t) := v^\top \big(x^{x_0,d_1}(t) - x^{x_0,d_2}(t)\big).$$

From (4.16) and $A^\top$-invariance of $\mathcal{V}^\perp$, we get

$$\frac{d^k \zeta}{dt^k}(t) = v^\top A^k \big(x^{x_0,d_1}(t) - x^{x_0,d_2}(t)\big)$$

for $k \geqslant 0$. The Cayley-Hamilton theorem implies that there exist real numbers c_i with $i = 0, 1, \ldots, n-1$ such that

$$\frac{d^{n_x}\zeta}{dt^{n_x}}(t) + c_{n_x-1}\frac{d^{n_x-1}\zeta}{dt^{n_x-1}}(t) + \cdots + c_1\frac{d\zeta}{dt}(t) + c_0\zeta(t) = 0.$$

Since

$$\frac{d^k \zeta}{dt^k}(0) = 0$$

for $k \geqslant 0$, we get $\zeta(t) = 0$ for all $t \geqslant 0$. Consequently, we have

$$x^{x_0,d_1}(t) - x^{x_0,d_2}(t) \in (\mathcal{V}^\perp)^\perp = \mathcal{V} = \langle A \mid \text{im } \begin{bmatrix} B & E \end{bmatrix} \rangle$$

which completes the proof. ∎

4.4 CONCLUSIONS

In this chapter we studied a class of nonsmooth and nonlinear dynamical systems, namely linear complementarity systems of index zero. These systems belong to the larger family of linear multi-modal systems, and are closely related to the piecewise affine dynamical systems in Chapter 3, for which the disturbance decoupling problem has already been solved. In this chapter we have shown that the linear subsystems of a linear complementarity system share certain geometric structure. By exploiting this geometric structure, we provided a necessary and sufficient condition for a linear complementarity system to be disturbance decoupled. Compared to the conditions for general piecewise affine systems in Chapter 3, this condition is crisper and more insightful.

Future research possibilities are weakening the technical blanket assumptions and studying disturbance decoupling problem under different feedback schemes. In the next chapter we will study the disturbance decoupling problem for a general class of linear multi-modal systems, and as a special case we study another class of linear complementarity systems, namely passive-like LCSs.

5

DISTURBANCE DECOUPLED LINEAR MULTI-MODAL SYSTEMS

ABSTRACT: *In this chapter we introduce the general framework of linear multi-modal systems and study the question under which conditions such a system is disturbance decoupled. We establish necessary conditions and sufficient conditions, both geometric in nature, from which almost all existing results on disturbance decoupledness for bimodal systems (Chapter 2), conewise linear systems (Chapter 3), linear complementarity systems of index zero (Chapter 4) and a particular class of switched linear systems can be recovered as special cases. Furthermore, we use this result to find novel conditions for disturbance decoupledness of a class of passive-like linear complementarity systems. This chapter is based on the journal paper [Everts and Camlibel, 2016], dedicated to the memory of J.C. Willems.*

5.1 INTRODUCTION

Annihilating or reducing the effects of disturbances is of major importance virtually in every real-life control problem. Designing feedback laws that decouple the disturbances from a certain to-be-controlled output constitute the well-known disturbance decoupling problem. The study of this problem for linear systems led to the development of geometric control theory [Basile and Marro, 1969a,b; Wonham and Morse, 1970] which provided solutions to numerous control problems as well as a deep understanding of the dynamics of linear systems [Wonham, 1985; Basile and Marro, 1992; Trentelman et al., 2001] and (smooth) nonlinear systems [Nijmeijer and van der Schaft, 1990; Isidori, 1995].

In this chapter, we focus on a class of hybrid dynamical systems and provide necessary and sufficient geometric conditions under which these systems are disturbance decoupled. Within the hybrid systems, the results on disturbance decoupling problem so far are limited to jumping hybrid systems [Conte et al., 2015], switched linear systems [Conte et al., 2014; Otsuka, 2010, 2011, 2015; Yurtseven et al., 2012; Zattoni et al., 2016; Zattoni and Marro, 2013], bimodal linear systems (Chapter 2), continuous piecewise affine systems 3 and a class of linear complementarity systems of index zero (Chapter 4). The results presented in these papers and chapters very much resemble those for the linear systems although their derivation is much harder, in particular, in the presence of state-dependent switching.

Within this chapter, we try to generalize these results by introducing the general framework of linear multi-modal systems, which contains the so-called conewise linear systems, linear complementarity systems [Heemels, 1999; Camlibel, 2001], and a particular class of switched linear systems [Sun and Ge, 2005; Liberzon, 2003] (as well as combinations of these) as particular cases. Later, we investigate necessary and sufficient conditions for a general linear multi-modal system to be disturbance decoupled. In addition, we show that almost all the existing results for the hybrid systems mentioned above can be recovered from the presented results as special cases.

Furthermore, we study a class of passive-like linear complementarity systems in detail in order to find novel necessary and sufficient conditions for this kind of systems to be disturbance decoupled.

The organization of this chapter is as follows. We introduce the framework of general linear multi-modal systems in Section 5.2 and discuss a few special cases. In Section 5.3 we define the property of being disturbance decoupled for a linear multi-modal system. We present our main results in Theorem 5.8 and in Theorem 5.9, which give a necessary condition and a sufficient condition for a linear multi-modal system to be disturbance decoupled. In Corollary 5.11 we show that in some cases these conditions coincide. We apply these results in Section 5.4 to the special cases introduced in Section 5.2. For one type of linear complementarity systems, this will lead to novel results, stated in Theorem 5.14. The chapter closes with the conclusions and discussions of possible future work in Section 5.5.

5.2 LINEAR MULTI-MODAL SYSTEMS

In this chapter we consider linear multi-modal systems given by the differential inclusion

$$\dot{x}(t) \in Ax(t) + Ed(t) + \Phi(y(t)) \tag{5.1a}$$

$$y(t) = Cx(t) + Fd(t) \tag{5.1b}$$

$$z(t) = Jx(t) \tag{5.1c}$$

where x is the state, d is the disturbance, y is the selection output, z is the to-be-controlled output, A, C, E, F and J are matrices of appropriate sizes and $\Phi : \mathbb{R}^{n_y} \to \mathbb{R}^{n_x}$ is a set-valued map satisfying

$$\Phi(y) = \{M_i y \mid i \in \mathcal{I} \text{ s.t. } y \in \mathcal{Y}_i\},$$

where $\mathcal{I}$ is a finite index set, $\{\mathcal{Y}_i\}_{i \in \mathcal{I}}$ is a collection of cones in $\mathbb{R}^{n_y}$, and $\{M_i\}_{i \in \mathcal{I}}$ is a collection of $n_x \times n_y$ matrices. The cones $\mathcal{Y}_i$ are not necessarily *solid* (i.e., n_y-dimensional). Moreover, the cones may overlap and their union does not have to be equal to $\mathbb{R}^{n_y}$. Without loss of generality we can assume that the matrix $\begin{bmatrix} C & F \end{bmatrix}$ has full row rank.

Let $T > 0$. For a given initial state x_0 and an integrable disturbance d we call an absolutely continuous function $x : [0, T) \to \mathbb{R}^{n_x}$ a solution on $[0, T)$ of system (5.1) if (5.1a) holds for almost all $t \in [0, T)$ and $x(0) = x_0$. If $T = +\infty$, we simply say that x is a (complete) solution of (5.1). In the sequel, we will allow multiple solutions for a given initial state and disturbance but make two assumptions regarding the existence of solutions.

The first assumption we make is that *local* solutions can be extended to complete solutions.

Assumption 5.1 If the system (5.1) admits a local solution x_T on $[0, T)$ for some $T > 0$, initial state x_0, and disturbance d, then there exists a complete solution x for the same initial state x_0 and disturbance satisfying $x(t) = x_T(t)$ for all $t \in [0, T)$.

The second assumption regarding the existence of solutions requires that the disturbances are not restricted by the dynamics of the system.

Assumption 5.2 If the system (5.1) admits a complete solution for some initial state and disturbance, then there exists a complete solution for the same initial state and for any disturbance.

Later on, we will elaborate on these assumptions when we discuss specific classes of systems that fall into the framework of (5.1).

We say that an initial state is *feasible* if for all locally integrable disturbances d there exists a complete solution of (5.1). The set of all feasible states will be denoted by $\mathcal{X}_0$.

To simplify the notation, we define

$$A_i = A + M_i C, \quad E_i = E + M_i F \tag{5.2}$$

and rewrite system (5.1) as

$$\dot{x}(t) \in \{A_i x(t) + E_i d(t) \mid i \in \mathcal{I} \text{ s.t. } y(t) \in \mathcal{Y}_i\} \tag{5.3a}$$
$$y(t) = Cx(t) + Fd(t) \tag{5.3b}$$
$$z(t) = Jx(t). \tag{5.3c}$$

We will work mainly with this form of the linear multi-modal system in the rest of the chapter.

Examples of systems that fall into this framework include switched linear systems, conewise linear systems, and linear complementarity problems, which we discuss next.

Example 5.3 (Switched Linear Systems) We consider the following particular class of linear switched systems

$$\dot{x}(t) = A_{\sigma(t)} x(t) + E_{\sigma(t)} d(t) \tag{5.4a}$$
$$z(t) = Jx(t), \tag{5.4b}$$

where σ is a switching signal from $\mathbb{R}_{\geqslant 0}$ to a finite index set $\mathcal{I}$. By taking $A = A_j$ and $E = E_j$ for some $j \in \mathcal{I}$, we can rewrite (5.4) in the form of a multi-modal system as

$$\dot{x}(t) \in Ax(t) + Ed(t) + \Phi(y) \tag{5.5a}$$
$$y(t) = \operatorname{col}(x(t), d(t)), \tag{5.5b}$$
$$z(t) = Jx(t), \tag{5.5c}$$

with

$$\Phi(y) = \{\begin{bmatrix} A_i - A & E_i - E \end{bmatrix} y \mid i \in \mathcal{I} \text{ s.t. } y \in \mathcal{Y}_i\}$$

and $\mathcal{Y}_i = \mathbb{R}^{n_x}$ for all i. Note that Assumptions 5.1 and 5.2 naturally hold for switched linear systems and $\mathcal{X}_0 = \mathbb{R}^{n_x}$.

Example 5.4 (Conewise Linear Systems) We say that a *continuous* function $\Phi : \mathbb{R}^{n_y} \to \mathbb{R}^{n_x}$ is *conewise linear* if there exist a finite family of solid polyhedral cones $\{\mathcal{Y}_i\}_{i\in\mathcal{I}}$ with $\cup_{i\in\mathcal{I}} \mathcal{Y}_i = \mathbb{R}^{n_y}$ and $n_x \times n_y$ matrices $\{M_i\}_{i\in\mathcal{I}}$ such that $g(y) = M_i y$ for $y \in \mathcal{Y}_i$.

Consider systems of the form

$$\dot{x}(t) = Ax(t) + Ed(t) + \Phi(y(t)) \tag{5.6a}$$
$$y(t) = Cx(t) + Fd(t) \tag{5.6b}$$
$$z(t) = Jx(t) \tag{5.6c}$$

where $\Phi : \mathbb{R}^{n_y} \to \mathbb{R}^{n_x}$ is a continuous conewise linear function. These systems will be called *conewise linear systems* (CLS) and were studied in [Camlibel et al., 2006; Arapostathis and Broucke, 2007; Camlibel et al., 2008]. CLSs can be seen as a special case of piecewise affine systems (Chapter 3) and fall naturally into the framework of (5.1). As the union of the (solid) cones $\mathcal{Y}_i$ is the entire $\mathbb{R}^{n_y}$, Assumptions 5.1 and 5.2 are satisfied and $\mathcal{X}_0 = \mathbb{R}^{n_x}$.

Example 5.5 (Complementarity Systems) We consider the linear complementarity system (LCS)

$$\dot{x}(t) = Ax(t) + Ed(t) + G\zeta(t) \tag{5.7a}$$
$$\eta(t) = Nx(t) + Rd(t) + H\zeta(t) \tag{5.7b}$$
$$0 \leqslant \zeta(t) \perp \eta(t) \geqslant 0 \tag{5.7c}$$
$$z(t) = Jx(t), \tag{5.7d}$$

where $\zeta, \eta \in \mathbb{R}^{n_\eta}$ are the so-called complementarity variables and all involved matrices are of appropriate dimensions.

Here, the inequalities for vectors are componentwise inequalities and $\perp$ denotes orthogonality. Linear complementarity systems are encountered in applications from various areas of engineering as well as operations research [van der Schaft and Schumacher, 2000; Heemels and Brogliato, 2003; Schumacher, 2004; Vasca et al., 2009]. For the work on the analysis and control of linear complementarity systems, we refer to [Heemels et al., 2000, 2002; Camlibel et al., 2002, 2003; Camlibel, 2007; Han et al., 2009; Heemels et al., 2011; Camlibel et al., 2014].

In this chapter, we focus on two particular classes of linear complementarity systems that were heavily studied in the literature:

1. H is a P-matrix, that is a matrix whose principal minors are all positive (see e.g. [Cottle et al., 1992]).
2. $R = 0$, $H = 0$, and NG is a symmetric positive definite matrix.

In what follows we will briefly derive the corresponding linear multi-modal systems for these two cases by skipping technical details for which we refer to [Heemels et al., 2000] for the first case and [Camlibel et al., 2014] for the second.

In the case that H is a P-matrix, the LCS (5.7) is of index zero and boils down to the multi-modal system

$$\dot{x}(t) \in \{A_\alpha x(t) + E_\alpha d(t) \mid \alpha \in \mathcal{I} \text{ s.t. } y(t) \in \mathcal{Y}_\alpha\} \tag{5.8a}$$
$$y(t) = Cx(t) + Fd(t) \tag{5.8b}$$
$$z(t) = Jx(t) \tag{5.8c}$$

where $\mathcal{I}$ is the set of all subsets of $\mathcal{I}_{n_\eta} = \{1, 2, \ldots, n_\eta\}$, and

$$\begin{aligned}
C &= N \\
F &= R \\
A_\alpha &= A - G_{\bullet\alpha}\left(H_{\alpha\alpha}\right)^{-1} N_{\alpha\bullet} \\
E_\alpha &= E - G_{\bullet\alpha}\left(H_{\alpha\alpha}\right)^{-1} N_{\alpha\bullet} \\
\mathcal{Y}_\alpha &= \{y \in \mathbb{R}^{n_\eta} \mid \begin{bmatrix} -(H_{\alpha\alpha})^{-1} I_{\alpha\bullet} \\ I_{\alpha^c\bullet} - H_{\alpha^c\alpha}(H_{\alpha\alpha})^{-1} I_{\alpha\bullet} \end{bmatrix} y \geqslant 0\}.
\end{aligned}$$

Note that in the above on the left-hand side the subscript α is used as an index, whereas on the right-hand side the subscript $\alpha\beta$ selects rows α and columns β of a matrix, for given index sets α and β. Here, the $\bullet$ means selecting all rows or columns and α^c denotes the complement of α in $\mathcal{I}_{n_\eta}$.

In the case that $R = 0$, $H = 0$, and NG is a symmetric positive definite matrix, the LCS (5.7) is passifiable by pole-shifting (see Definition 3.4.2 in [Camlibel, 2001] and [Camlibel et al., 2014]) and boils down to the multi-modal system (5.8) where $\alpha \subseteq \mathcal{I}_{n_\eta}$ and

$$\begin{aligned}
C &= \begin{bmatrix} N \\ NA \end{bmatrix} \\
F &= \begin{bmatrix} 0 \\ NE \end{bmatrix} \\
A_\alpha &= A - G_{\bullet\alpha}(N_{\alpha\bullet}G_{\bullet\alpha})^{-1} N_{\alpha\bullet} A \\
E_\alpha &= E - G_{\bullet\alpha}(N_{\alpha\bullet}G_{\bullet\alpha})^{-1} N_{\alpha\bullet} E \\
\mathcal{Y}_\alpha &= \{y \in \mathbb{R}^{2n_\eta} \mid \begin{bmatrix} I_{\alpha^c\bullet} & 0 \\ 0 & -(N_{\alpha\bullet}G_{\bullet\alpha})^{-1} I_{\alpha\bullet} \end{bmatrix} y \geqslant 0, \\
&\qquad \begin{bmatrix} I_{\alpha\bullet} & 0 \end{bmatrix} y = 0\}.
\end{aligned}$$

For both cases, Assumptions 5.1 and 5.2 are satisfied [Heemels et al., 2000; Camlibel et al., 2014]. We have $\mathcal{X}_0 = \mathbb{R}^{n_x}$ (see e.g. [Heemels et al., 2000]) in case H is a P-matrix and $\mathcal{X}_0 = \{x_0 \mid Cx_0 \geqslant 0\}$ for the second case (see e.g. [Camlibel et al., 2014]).

5.3 DISTURBANCE DECOUPLED SYSTEMS

We start with the following definition of a disturbance decoupled system.

Definition 5.6 We say that system (5.3) is *disturbance decoupled* if for any given feasible initial state $x_0 \in \mathcal{X}_0$ and any two solutions $(x_1(t), y_1(t), z_1(t))$ and $(x_2(t), y_2(t), z_2(t))$, corresponding to any two locally integrable disturbances $d_1(t)$ and $d_2(t)$ respectively, satisfy

$$z_1(t) = z_2(t)$$

for all $t \geqslant 0$.

In this chapter we investigate when system (5.3) is disturbance decoupled. Throughout the chapter we assume the following.

Assumption 5.7 For each $i \in \mathcal{I}$, the cone $\mathcal{Y}_i$ and the subspace $\operatorname{im} F + C\langle A_i \mid \operatorname{im} E_i \rangle$ satisfy

i. $\operatorname{im} F + C\langle A_i \mid \operatorname{im} E_i \rangle \subseteq \operatorname{span}(\mathcal{Y}_i)$,

ii. $\big(\operatorname{im} F + C\langle A_i \mid \operatorname{im} E_i \rangle\big) \cap \operatorname{rint}(\mathcal{Y}_i) \neq \varnothing$, or $\mathcal{Y}_i$ is solid.

The first assumption is trivial when each cone $\mathcal{Y}_i$ is solid. A consequence of this assumption is that $\operatorname{im} F \subseteq \cap_{i \in \mathcal{I}} \operatorname{span}(\mathcal{Y}_i)$. The second assumption assures a certain 'liveliness' of each cone $\mathcal{Y}_i$; for every cone $\mathcal{Y}_i$ there exist a point x_0 and a locally integrable disturbance $d(t)$ such that $y^{x_0,d}(t)$ stays in $\operatorname{rint}(\mathcal{Y}_i)$ for some time t. If $F + C(sI - A)^{-1}E$ is right invertible, using (1.9), (1.6), and (1.7), one can see that $\operatorname{im} F + C\langle A_i \mid \operatorname{im} E_i \rangle = \mathbb{R}^{n_y}$, which implies the second assumption.

A necessary condition for a linear multi-modal system to be disturbance decoupled is stated in the following theorem.

Theorem 5.8 *If a linear multi-modal system of the form* (5.3)*, satisfying Assumptions* 5.1*,* 5.2 *and* 5.7*, is disturbance decoupled, then*

$$\sum_{i \in \mathcal{I}} \langle A_i \mid \operatorname{im} E_i \rangle \subseteq \ker J. \tag{5.9}$$

Proof. Fix $i \in \mathcal{I}$. Since $\begin{bmatrix} C & F \end{bmatrix}$ is of full row rank, there exist an $x_0 \in \mathbb{R}^{n_x}$ and a $d \in \mathbb{R}^{n_d}$ such that

$$y_0 := Cx_0 + Fd \in \operatorname{rint}(\mathcal{Y}_i).$$

If the first condition in Assumption 5.7 holds, then we can even pick $x_0 \in \langle A_i \mid \operatorname{im} E_i \rangle$. Consider the solution $(\tilde{x}(t), \tilde{y}(t))$ of the following linear system

$$\dot{\tilde{x}}(t) = A_i \tilde{x}(t) + E_i d(t) \tag{5.10a}$$
$$\tilde{y}(t) = C\tilde{x}(t) + Fd(t), \tag{5.10b}$$

where $d(t) = d$ and with $\tilde{x}(0) = x_0$. If $\mathcal{Y}_i$ is solid, then the continuity of $\tilde{y}(t)$ implies that there exists an $\varepsilon > 0$ such that $\tilde{y}(t) \in \operatorname{rint}(\mathcal{Y}_i)$ for all $t \in [0, \varepsilon]$. In the case that we have $x_0 \in \langle A_i \mid \operatorname{im} E_i \rangle$, we see that $\tilde{x}(t) \in \langle A_i \mid \operatorname{im} E_i \rangle$ for all $t \geqslant 0$. Hence, by Assumption 5.7(i),

$$\tilde{y}(t) \in \operatorname{im} F + C\langle A_i \mid \operatorname{im} E_i \rangle \subseteq \operatorname{span}(\mathcal{Y}_i)$$

for all $t \geqslant 0$. Since $\tilde{y}(t)$ is continuous, and $\tilde{y}(0) = y_0 \in \operatorname{rint}(\mathcal{Y}_i)$, it follows that there again exists an $\varepsilon > 0$ such that $\tilde{y}(t) \in \operatorname{rint}(\mathcal{Y}_i)$ for all $t \in [0, \varepsilon]$.

Let e be any vector in $\mathbb{R}^{n_d}$, then we have

$$Cx_0 + F(d + \mu e) = y_0 + F\mu e \in \operatorname{span}(\mathcal{Y}_i),$$

for any $\mu \in \mathbb{R}$, since $\operatorname{im} F \subseteq \operatorname{span}(\mathcal{Y}_i)$ by Assumption 5.7(i). By taking $|\mu|$ sufficiently small, we have $Cx_0 + F(d + \mu e) \in \operatorname{rint}(\mathcal{Y}_i)$. Let $\tilde{x}_e(t)$ be the solution of (5.10) for the constant disturbance $d_e(t) = d + \mu e$ and initial condition $\tilde{x}_e(0) = x_0$, with corresponding output $\tilde{y}_e(t)$. For $\tilde{y}_e(t)$ there is an $\varepsilon_e > 0$ such that $\tilde{y}_e(t) \in \operatorname{rint}(\mathcal{Y}_i)$ for $t \in [0, \varepsilon_e]$.

Let $\varepsilon^* = \min(\varepsilon, \varepsilon_e)$. Due to Assumption 5.1, we can extend $\tilde{x}(t)$ and $\tilde{x}_e(t)$ from $t = \varepsilon^*$ onwards to obtain complete solutions $x(t)$ and $x_e(t)$ of system (5.3), with corresponding outputs $(y(t), z(t))$ and $(y_e(t), z_e(t))$, respectively. Moreover, $x_0 \in \mathcal{X}_0$ due to Assumption 5.2.

Since system (5.3) is disturbance decoupled, we have that

$$z(t) - z_e(t) = J(x(t) - x_e(t)) = 0 \tag{5.11}$$

for all $t \geqslant 0$. Since $d(t)$ and $d_e(t)$ are constant, we can differentiate (5.11) repeatedly and evaluate at $t = 0$ to obtain

$$JA_i^k E_i \mu e = 0$$

for all $k \geqslant 0$. Since this holds for all $e \in \mathbb{R}^{n_d}$, we have $JA_i^k E_i = 0$ for all k. Consequently, by (1.2), we have $\langle A_i \mid \operatorname{im} E_i \rangle \subseteq \ker J$. As this holds for all $i \in \mathcal{I}$, we can conclude that (5.9) holds. ■

Next we give a sufficient condition for a linear multi-modal system to be disturbance decoupled. For this purpose, we define the subspaces

$$\mathcal{A} := \sum_{i,j\in\mathcal{I}} \operatorname{im}(A_j - A_i), \quad \mathcal{E} := \sum_i \operatorname{im} E_i. \tag{5.12}$$

Theorem 5.9 *If there is a subspace $\mathcal{V} \subseteq \ker J$ such that $A_i\mathcal{V} \subseteq \mathcal{V}$ for each $i \in \mathcal{I}$, $\mathcal{E} \subseteq \mathcal{V}$ and $\mathcal{A} \subseteq \mathcal{V}$, then the linear multi-modal system (5.3) satisfying Assumption 5.7 is disturbance decoupled.*

Proof. Let $x_0 \in \mathcal{X}_0$ be any given feasible initial state, and let $n_v = \dim \mathcal{V}$. Furthermore, let $\{\xi_1, \xi_2, \ldots, \xi_{n_x}\}$ be a basis for $\mathbb{R}^{n_x}$ such that $\{\xi_1, \xi_2, \ldots, \xi_{n_v}\}$ forms a basis for $\mathcal{V}$. With respect to these coordinates, we can write every $x \in \mathbb{R}^{n_x}$ uniquely as $x = \operatorname{col}(v, w)$ for some $v \in \mathbb{R}^{n_v}$ and $w \in \mathbb{R}^{n_x - n_v}$ such that $\operatorname{col}(v, 0) \in \mathcal{V}$. Since $\mathcal{V}$ is A_i-invariant for each $i \in \mathcal{I}$ and $\mathcal{E} \subseteq \mathcal{V} \subseteq \ker J$, with respect to the new coordinates we have

$$A_i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ 0 & A_{22}^i \end{bmatrix}, \quad E_i = \begin{bmatrix} E_1^i \\ 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & J_2 \end{bmatrix},$$

for every $i \in \mathcal{I}$, where $A_{11}^i \in \mathbb{R}^{n_v \times n_v}$, $E_1^i \in \mathbb{R}^{n_v \times n_d}$ and $J_2 \in \mathbb{R}^{(n_x - n_v) \times n_x}$. Let $x_0 = \operatorname{col}(v_0, w_0)$, and let $d(t)$ be any locally integrable disturbance. Write $x(t) = \operatorname{col}(v(t), w(t))$, then $v(t)$ and $w(t)$ satisfy

$$\begin{aligned} \dot{v}(t) \in \{ & A_{11}^i v(t) + A_{12}^i w(t) + E_1^i d(t) \mid \\ & \text{for } i \in \mathcal{I} \text{ s.t. } C \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} + F d(t) \in \mathcal{Y}_i \} \\ \dot{w}(t) &= A_{22}^i w(t) \\ z(t) &= J_2 w(t) \end{aligned}$$

for almost all t, with $v(0) = v_0$ and $w(0) = w_0$. Since $\operatorname{im}(A_j - A_i) \subseteq \mathcal{V}$ for all $i, j \in \mathcal{I}$, we have $A_{22}^i = A_{22}^j$ for all $i, j \in \mathcal{I}$. Therefore, $w(t)$ will satisfy the linear differential equation

$$\dot{w}(t) = A_{22}^i w(t), \quad w(0) = w_0$$

for any fixed $i \in \mathcal{I}$ and almost all t. We see that $w(t)$ does not depend on the disturbance $d(t)$. Since the output z satisfies $z(t) = J_2 w(t)$, we see that z does not depend on the disturbance either. Hence, system (5.3) is disturbance decoupled. ■

The subspace $\sum_{i\in\mathcal{I}} \langle A_i \mid \operatorname{im} E_i \rangle$ plays an important role in our main results. Although each subspace $\langle A_j \mid \operatorname{im} E_j \rangle$ is invariant under A_j, their sum $\sum_{i\in\mathcal{I}} \langle A_i \mid \operatorname{im} E_i \rangle$ is not necessarily invariant under each A_j, so it is not always possible to use $\sum_{i\in\mathcal{I}}\langle A_i \mid \operatorname{im} E_i\rangle$ as the subspace $\mathcal{V}$ in Theorem 5.9. In the next lemma we give some conditions under which the subspace $\sum_{i\in\mathcal{I}}\langle A_i \mid \operatorname{im} E_i\rangle$ is A_i-invariant for each $i \in \mathcal{I}$ and has a more compact form.

Lemma 5.10 *Let A_i and E_i satisfy (5.2). The subspace $\sum_{i\in\mathcal{I}}\langle A_i \mid \operatorname{im} E_i\rangle$ is A_j-invariant and satisfies*

$$\sum_{i\in\mathcal{I}}\langle A_i \mid \operatorname{im} E_i\rangle = \langle A_j \mid \mathcal{A} + \mathcal{E}\rangle \tag{5.13}$$

for each $j \in \mathcal{I}$ if one of the following conditions holds:

i. $\mathcal{A} \subseteq \sum_{i\in\mathcal{I}}\langle A_i \mid \operatorname{im} E_i\rangle$,

ii. $\operatorname{im}(M_j - M_i) \subseteq \sum_{i\in\mathcal{I}}\langle A_i \mid \operatorname{im} E_i\rangle$ *for all* $i, j \in \mathcal{I}$,

iii. $(M_j - M_i)(\operatorname{im} F + C\mathcal{T}^*(A, E, C, F)) = \operatorname{im}(M_j - M_i)$ *for all* $i, j \in \mathcal{I}$,

iv. $F + C(sI - A)^{-1}E$ *is right invertible.*

Proof. We will prove this lemma by showing that *iv.* $\Rightarrow$ *iii.* $\Rightarrow$ *ii.* $\Rightarrow$ *i.* $\Rightarrow$ (5.13). Define

$$\mathcal{V} := \sum_{i\in\mathcal{I}}\langle A_i \mid \operatorname{im} E_i\rangle, \quad \mathcal{T}^* := \mathcal{T}^*(A, E, C, F).$$

(*iv.* $\Rightarrow$ *iii.*) If $F + C(sI - A)^{-1}E$ is right invertible, then using (1.9) we find that $\operatorname{im} F + C\mathcal{T}^* = \mathbb{R}^{n_y}$, which implies the third condition.

(*iii.* $\Rightarrow$ *ii.*) From (1.6), we see that the subspace $\mathcal{T}^*$ satisfies $\mathcal{T}^* = \mathcal{T}^*(A_i, E_i, C, F)$ for each $i \in \mathcal{I}$. Using (1.7) this gives us

$$\mathcal{T}^* \subseteq \langle A_i \mid \operatorname{im} E_i\rangle,$$

for each $i \in \mathcal{I}$, which implies

$$A_i\mathcal{T}^* \subseteq A_i\langle A_i \mid \operatorname{im} E_i\rangle \subseteq \langle A_i \mid \operatorname{im} E_i\rangle \subseteq \mathcal{V}.$$

Furthermore, we have that

$$\operatorname{im} E_i \subseteq \langle A_i \mid \operatorname{im} E_i\rangle \subseteq \mathcal{V}$$

for all i in $\mathcal{I}$. This yields

$$\begin{aligned}(M_j - M_i)C\mathcal{T}^* &= (A_j - A_i)\mathcal{T}^* \subseteq \mathcal{V}\\ \operatorname{im}(M_j - M_i)F &= \operatorname{im}(E_j - E_i) \subseteq \mathcal{V}\end{aligned}$$

for any $i, j \in \mathcal{I}$. Together, this gives us

$$(M_j - M_i)(\operatorname{im} F + C\mathcal{T}^*) \subseteq \mathcal{V}.$$

From the third condition it follows that $\operatorname{im}(M_j - M_i) \subseteq \mathcal{V}$ for all $i, j \in \mathcal{I}$.

($ii. \Rightarrow i.$) This follows from the fact that $\operatorname{im}(A_j - A_i) = \operatorname{im}(M_j - M_i)C \subseteq \operatorname{im}(M_j - M_i)$.

($i. \Rightarrow$ (5.13)) For any $i, j \in \mathcal{I}$ we have

$$\begin{aligned} A_j\langle A_i \mid \operatorname{im} E_i\rangle &\subseteq A_i\langle A_i \mid \operatorname{im} E_i\rangle \\ &\quad + (A_j - A_i)\langle A_i \mid \operatorname{im} E_i\rangle \\ &\subseteq \langle A_i \mid \operatorname{im} E_i\rangle + \operatorname{im}(A_j - A_i) \\ &\subseteq \mathcal{V}, \end{aligned}$$

where we used $\mathcal{A} \subseteq \mathcal{V}$ in the last step. Hence, we see that

$$\begin{aligned} A_j\mathcal{V} &= A_j\Big(\sum_{i\in\mathcal{I}}\langle A_i \mid \operatorname{im} E_i\rangle\Big) \\ &\subseteq \sum_{i\in\mathcal{I}} A_j\langle A_i \mid \operatorname{im} E_i\rangle \subseteq \mathcal{V}, \end{aligned}$$

thus $\mathcal{V}$ is A_j-invariant for every $j \in \mathcal{I}$. Since $\operatorname{im} E_i \subseteq \mathcal{V}$ for all $i \in \mathcal{I}$ it follows that $\mathcal{E} \subseteq \mathcal{V}$, and hence $\mathcal{A} + \mathcal{E} \subseteq \mathcal{V}$. Consequently,

$$\langle A_j \mid \mathcal{A} + \mathcal{E}\rangle \subseteq \mathcal{V}, \tag{5.14}$$

since $\langle A_j \mid \mathcal{A} + \mathcal{E}\rangle$ is the smallest A_j-invariant subspace containing $\mathcal{A} + \mathcal{E}$.

For the other inclusion, note that

$$\begin{aligned} A_i\langle A_j \mid \mathcal{A} + \mathcal{E}\rangle &\subseteq A_j\langle A_j \mid \mathcal{A} + \mathcal{E}\rangle \\ &\quad + (A_i - A_j)\langle A_j \mid \mathcal{A} + \mathcal{E}\rangle \\ &\subseteq \langle A_j \mid \mathcal{A} + \mathcal{E}\rangle + \mathcal{A} \\ &\subseteq \langle A_j \mid \mathcal{A} + \mathcal{E}\rangle, \end{aligned}$$

which means that $\langle A_j \mid \mathcal{A} + \mathcal{E}\rangle$ is A_i-invariant for all $i \in \mathcal{I}$. Furthermore, we have $\operatorname{im} E_i \subseteq \langle A_j \mid \mathcal{A} + \mathcal{E}\rangle$ for each $i \in \mathcal{I}$. Since $\langle A_i \mid \operatorname{im} E_i\rangle$ is the smallest A_i-invariant subspace containing $\operatorname{im} E_i$, we see that

$$\langle A_i \mid \operatorname{im} E_i\rangle \subseteq \langle A_j \mid \mathcal{A} + \mathcal{E}\rangle,$$

for each $i \in \mathcal{I}$, and hence $\mathcal{V} \subseteq \langle A_j \mid \mathcal{A} + \mathcal{E} \rangle$. Together with (5.14) this completes the proof. ∎

If one of the conditions in Lemma 5.10 is satisfied, we can combine Theorems 5.8 and 5.9 to obtain the following necessary and sufficient conditions for system (5.3) to be disturbance decoupled.

Corollary 5.11 *Assume that Assumptions 5.1, 5.2, and 5.7 are satisfied. If one of the conditions in Lemma 5.10 holds, then the linear multi-modal system (5.3) is disturbance decoupled if and only if*

$$\langle A_j \mid \mathcal{A} + \mathcal{E} \rangle \subseteq \ker J,$$

for every $j \in \mathcal{I}$.

5.4 SPECIAL CLASSES OF SYSTEMS

In this section we revisit the examples discussed in Section 5.2 and apply Theorem 5.8, Theorem 5.9, and Corollary 5.11 to these systems. For the linear complementarity problem with $R = 0$, $H = 0$, and NG a symmetric positive definite matrix this will lead to new results, which are presented in Section 5.4.3.2. For the switched linear systems, conewise linear systems and the other linear complementarity problem, we compare our result with existing results in the literature.

5.4.1 *Switched linear systems*

The disturbance decoupling problem for switched linear systems has been studied in [Yurtseven et al., 2012], in which a distinction is made between disturbance decoupling (DD) w.r.t. d and DD w.r.t. the switching signal σ. From Theorem 3.7 in [Yurtseven et al., 2012] we see that system (5.4) is disturbance decoupled (w.r.t. both d and σ) if and only if there exists a subspace $\mathcal{V}$ that is invariant under all A_i, satisfying

$$\operatorname{im}(A_i - A_j) \subseteq \mathcal{V} \subseteq \ker J, \qquad \operatorname{im} E_i \subseteq \mathcal{V},$$

for all $i, j \in \mathcal{I}$.

The switched linear system satisfies Assumption 5.7 since every $\mathcal{Y}_i$ equals $\mathbb{R}^{n_y}$. Therefore, we can apply Theorem 5.9, which gives the same sufficient condition as above for system (5.4) to be disturbance decoupled. However, the necessary condition we get from Theorem 5.8 is slightly weaker. This discrepancy can be explained by the observation that for switched linear systems the relative interior of every two cones $\mathcal{Y}_i$ and $\mathcal{Y}_j$ intersect, which means that for each $i, j \in \mathcal{I}$ there is an open neighborhood in $\mathbb{R}^{n_y}$ in which the mode can change arbitrarily from i to j and back.

In the case that $\mathcal{A} \subseteq \sum_{i\in\mathcal{I}} \langle A_i \mid \operatorname{im} E_i \rangle$, we have that $\sum_{i\in\mathcal{I}} \langle A_i \mid \operatorname{im} E_i \rangle$ is the smallest subspace that contains $\operatorname{im} E_i$ and is A_i-invariant for each $i \in \mathcal{I}$, and hence we could take $\sum_{i\in\mathcal{I}} \langle A_i \mid \operatorname{im} E_i \rangle$ as the subspace $\mathcal{V}$ in Theorem 3.7 in [Yurtseven et al., 2012].

5.4.2 *Conewise linear systems*

Conewise linear systems can be seen as a special case of piecewise affine systems. In the case that $F = 0$, Corollary 3.4 in Chapter 3 shows that

$$\sum_{i\in\mathcal{I}} \langle A_i \mid \operatorname{im} E \rangle \subseteq \ker J \tag{5.15}$$

is a necessary condition for the conewise linear system (5.6) to be disturbance decoupled. Corollary 3.6 in Chapter 3 states that system (5.6) is disturbance decoupled if there is a subspace $\mathcal{V} \subseteq \ker J$ that contains $\operatorname{im} E$ and is invariant under each A_i. The necessary condition (5.15) can be recovered by Theorem 5.8, since Assumption 5.7 is satisfied, as each cone $\mathcal{Y}_i$ is solid. From Theorem 5.9 we find that the existence of a subspace $\mathcal{V} \subseteq \ker J$ that is invariant under each A_i and contains $\operatorname{im} E$ *and* $\mathcal{A}$ is a sufficient condition for system (5.6) to be disturbance decoupled. This condition is stronger than the condition in Corollary 3.6. This difference can be explained by the continuity assumption for the conewise linear system, which cannot be exploited for general linear multi-modal systems. In the case that $C(sI - A)^{-1}E$ is right invertible, then Corollary 3.10 in Chapter 3 yields (5.15) as a necessary and sufficient condition for disturbance decoupledness, which can be recovered by Corollary 5.11 in this chapter.

A bimodal linear system is a special case of conewise linear systems. We consider the case that $y = c^\top x$ for some vector c. We have shown in Chapter 2 that such a bimodal linear system is disturbance decoupled if and only if

$$\langle A_1 \mid \operatorname{im} E \rangle + \langle A_2 \mid \operatorname{im} E \rangle \subseteq \ker J,$$

even if $c^\mathsf{T}(sI-A)^{-1}E$ is not right invertible. For this particular system, the condition (i) of Lemma 5.10 holds regardless of whether $c^\mathsf{T}(sI-A)^{-1}E$ is right-invertible or not. As such, the necessary and sufficient conditions for bimodal systems in Chapter 2 can be recovered from Corollary 5.11.

5.4.3 *Linear complementarity systems*

5.4.3.1 *Case 1*

In Theorem 4.2 in Chapter 4 we have shown that linear complementarity system (5.7), with H being a P-matrix and the transfer matrix $R+N(sI-A)^{-1}E$ right invertible, is disturbance decoupled if and only if

$$\sum_{\alpha\in\mathcal{I}} \langle A_\alpha \mid \operatorname{im} E_\alpha\rangle \subseteq \ker J. \tag{5.16}$$

This result can be recovered from Corollary 5.11. To see this, note that the cones $\mathcal{Y}_i$ are solid for this linear complementarity system, which can be seen from the right-invertibility of $R+N(sI-A)^{-1}E$ and H being a P-matrix. Thus, Assumption 5.7 is satisfied, and since Assumptions 5.1 and 5.2 also hold (see Example 5.5), we can indeed apply Corollary 5.11 and find (5.16) as a necessary and sufficient condition for system (5.7) to be disturbance decoupled.

By exploiting the special relation between the matrices A_α and E_α, we have shown in Lemma 4.1 in Chapter 4 that

$$\sum_{\alpha\in\mathcal{I}} \langle A_\alpha \mid \operatorname{im} E_\alpha\rangle = \langle A \mid \operatorname{im} \begin{bmatrix} E & G\end{bmatrix}\rangle.$$

Therefore, we find that

$$\langle A \mid \operatorname{im} \begin{bmatrix} E & G\end{bmatrix}\rangle \subseteq \ker J.$$

is a necessary and sufficient geometric condition for system (5.7) to be disturbance decoupled.

5.4.3.2 *Case 2*

We consider again the Linear Complementarity System (5.7), and now we assume that $R=0$, $H=0$, NG is a symmetric positive definite matrix and that the transfer matrix $N(sI-A)^{-1}E$ is right invertible as a rational matrix.

It turns out that checking that system (5.7) satisfies Assumption 5.7 requires more effort than case 1, as the cones $\mathcal{Y}_i$ in this case are not all solid.

Lemma 5.12 *Suppose that $R = 0$, $H = 0$, NG is a symmetric positive definite matrix and that $N(sI - A)^{-1}E$ is right invertible. Then system (5.7) satisfies Assumption 5.7.*

Proof. We start with the observation that the right-invertibility of the transfer matrix $N(sI - A)^{-1}E$ implies that the transfer matrix

$$T_\alpha(s) := \begin{bmatrix} 0 \\ N_{\alpha\bullet}E \end{bmatrix} + \begin{bmatrix} N_{\alpha^c\bullet} \\ N_{\alpha\bullet}A \end{bmatrix} (sI - A)^{-1}E$$

is also right invertible for any $\alpha \subseteq \mathcal{I}_{n_\eta}$. Indeed, suppose that there is a rational vector $\mathrm{col}(u_{\alpha^c}(s), u_\alpha(s))$ such that

$$\begin{bmatrix} u_{\alpha^c}^{\mathsf{T}}(s) & u_\alpha^{\mathsf{T}}(s) \end{bmatrix} T_\alpha(s) = 0.$$

Then, from the relation

$$s \cdot N_{\alpha\bullet}(sI - A)^{-1}E = N_{\alpha\bullet}E + N_{\alpha\bullet}A(sI - A)^{-1}E,$$

we see that

$$\begin{aligned} &\begin{bmatrix} u_{\alpha^c}^{\mathsf{T}}(s) & \frac{1}{s} \cdot u_\alpha^{\mathsf{T}}(s) \end{bmatrix} T_\alpha(s) \\ &= \begin{bmatrix} u_{\alpha^c}^{\mathsf{T}}(s) & u_\alpha^{\mathsf{T}}(s) \end{bmatrix} \begin{bmatrix} N_{\alpha^c\bullet} \\ N_{\alpha\bullet} \end{bmatrix} (sI - A)^{-1}E = 0. \end{aligned}$$

The right-invertibility of the rational matrix $N(sI - A)^{-1}E$ implies that $\begin{bmatrix} u_{\alpha^c}^{\mathsf{T}}(s) & u_\alpha^{\mathsf{T}}(s) \end{bmatrix} = 0$. Consequently, $T_\alpha(s)$ is right invertible.

Next, we use (1.6) and (1.7) to observe that

$$\begin{aligned} &\mathcal{T}^*(A, E, \begin{bmatrix} N_{\alpha^c\bullet} \\ N_{\alpha\bullet}A \end{bmatrix}, \begin{bmatrix} 0 \\ N_{\alpha\bullet}E \end{bmatrix}) \\ &\qquad = \mathcal{T}^*(A_\alpha, E_\alpha, \begin{bmatrix} N_{\alpha^c\bullet} \\ N_{\alpha\bullet}A \end{bmatrix}, \begin{bmatrix} 0 \\ N_{\alpha\bullet}E \end{bmatrix}) \\ &\qquad \subseteq \langle A_\alpha \mid \mathrm{im}\, E_\alpha \rangle. \end{aligned}$$

Since $T_\alpha(s)$ is right invertible we can use this together with (1.9) to find that

$$\mathrm{im} \begin{bmatrix} 0 \\ N_{\alpha\bullet}E \end{bmatrix} + \begin{bmatrix} N_{\alpha^c\bullet} \\ N_{\alpha\bullet}A \end{bmatrix} \langle A_\alpha \mid \mathrm{im}\, E_\alpha \rangle = \mathbb{R}^{n_\eta}$$

or, equivalently,

$$\begin{bmatrix} N_{\alpha^c\bullet} & 0 \\ N_{\alpha\bullet}A & N_{\alpha\bullet}E \end{bmatrix} \left(\langle A_\alpha \mid \mathrm{im}\, E_\alpha \rangle \times \mathbb{R}^{n_d} \right) = \mathbb{R}^{n_\eta}. \tag{5.17}$$

We rewrite $\mathcal{Y}_\alpha$ as

$$\mathcal{Y}_\alpha = \begin{bmatrix} N & 0 \\ NA & NE \end{bmatrix} \tilde{\mathcal{Y}}_\alpha$$

with

$$\tilde{\mathcal{Y}}_\alpha = \{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} \geqslant 0 \} \cap \ker \begin{bmatrix} N_{\alpha\bullet} & 0 \end{bmatrix},$$

where

$$\Theta_\alpha = \begin{bmatrix} I & 0 \\ 0 & -(N_{\alpha\bullet} G_{\bullet\alpha})^{-1} \end{bmatrix} \begin{bmatrix} N_{\alpha^c\bullet} & 0 \\ N_{\alpha\bullet} A & N_{\alpha\bullet} E \end{bmatrix}. \tag{5.18}$$

From (5.17) we see that

$$\Theta_\alpha \big(\langle A_\alpha \mid \operatorname{im} E_\alpha \rangle \times \mathbb{R}^{n_d} \big) = \mathbb{R}^{n_\eta} \tag{5.19}$$

since the first matrix on the right-hand-side of (5.18) is non-singular. Hence Θ_α has full row rank, which gives us that

$$\operatorname{rint} \{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} \geqslant 0 \} = \{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} > 0 \}.$$

Furthermore, (5.17) also shows that

$$\{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} > 0 \} \cap \ker \begin{bmatrix} N_{\alpha\bullet} & 0 \end{bmatrix} \neq \varnothing. \tag{5.20}$$

Therefore, we can use Proposition 2.42 in [Rockafellar and Wets, 2009] to find that the relative interior of $\tilde{\mathcal{Y}}_\alpha$ is given by

$$\operatorname{rint} \tilde{\mathcal{Y}}_\alpha = \{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} > 0 \} \cap \ker \begin{bmatrix} N_{\alpha\bullet} & 0 \end{bmatrix}.$$

Note that

$$N_{\alpha\bullet} A_\alpha = 0, \qquad N_{\alpha\bullet} E_\alpha = 0, \tag{5.21}$$

and hence $\langle A_\alpha \mid \operatorname{im} E_\alpha \rangle \subseteq \ker N_{\alpha\bullet}$. Consequently, we have that

$$\begin{aligned} &(\langle A_\alpha \mid \operatorname{im} E_\alpha \rangle \times \mathbb{R}^{n_d}) \cap \operatorname{rint} \tilde{\mathcal{Y}}_\alpha \\ &= (\langle A_\alpha \mid \operatorname{im} E_\alpha \rangle \times \mathbb{R}^{n_d}) \cap \{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} > 0 \}, \end{aligned}$$

which is non-empty, due to (5.19). Hence, the set

$$\begin{bmatrix} N & 0 \\ NA & NE \end{bmatrix} \big(((\langle A_\alpha \mid \operatorname{im} E_\alpha \rangle \times \mathbb{R}^{n_d}) \cap \operatorname{rint} \tilde{\mathcal{Y}}_\alpha \big)$$

is also non-empty. Using Proposition 2.44(a) in [Rockafellar and Wets, 2009], this implies that

$$\left(\begin{bmatrix} N \\ NA \end{bmatrix} \langle A_\alpha \mid \operatorname{im} E_\alpha \rangle + \operatorname{im} \begin{bmatrix} 0 \\ NE \end{bmatrix}\right) \cap \operatorname{rint}(\mathcal{Y}_\alpha) \neq \varnothing,$$

which is Assumption 5.7(ii).

From (5.20) we know that there is a point $\bar{y}$ such that $\Theta_\alpha \bar{y} > 0$ and $\begin{bmatrix} N_{\alpha\bullet} & 0 \end{bmatrix} \bar{y} = 0$. This implies that for every $y \in \begin{bmatrix} N_{\alpha\bullet} & 0 \end{bmatrix}$ there is a $\gamma \in \mathbb{R}$ such that $y + \gamma\bar{y} \in \tilde{\mathcal{Y}}_\alpha$, so $y \in \operatorname{span}(\tilde{\mathcal{Y}}_\alpha)$. Hence, $\ker \begin{bmatrix} N_{\alpha\bullet} & 0 \end{bmatrix} \subseteq \operatorname{span}(\tilde{\mathcal{Y}}_\alpha)$. Together with $\tilde{\mathcal{Y}}_\alpha \subseteq \ker \begin{bmatrix} N_{\alpha\bullet} & 0 \end{bmatrix}$ this gives us

$$\operatorname{span}(\tilde{\mathcal{Y}}_\alpha) = \ker \begin{bmatrix} N_{\alpha\bullet} & 0 \end{bmatrix}.$$

With (5.21) this gives us

$$\langle A_\alpha \mid \operatorname{im} E_\alpha \rangle \times \mathbb{R}^{n_d} \subseteq \operatorname{span}(\tilde{\mathcal{Y}}_\alpha).$$

Hence,

$$\left(\begin{bmatrix} N \\ NA \end{bmatrix} \langle A_\alpha \mid \operatorname{im} E_\alpha \rangle + \operatorname{im} \begin{bmatrix} 0 \\ NE \end{bmatrix}\right) \subseteq \operatorname{span}(\mathcal{Y}_\alpha),$$

and hence system (5.7) also satisfies Assumption 5.7(i). ■

Lemma 5.10 cannot directly be applied to system (5.7), since the right-invertibility of $N(sI - A)^{-1}E$ does not imply that

$$\begin{bmatrix} 0 \\ NE \end{bmatrix} + \begin{bmatrix} N \\ NA \end{bmatrix} (sI - A)^{-1}E$$

is right invertible. However, the relation

$$N(sI - A)^{-1}E = \frac{1}{s}\left(NE + NA(sI - A)^{-1}E\right)$$

reveals that the right-invertibility of $N(sI - A)^{-1}E$ implies that $NE + NA(sI - A)^{-1}E$ is right invertible as well. So if we take $\tilde{C} = NA$ and $\tilde{F} = NE$ and write

$$A_\alpha = A + M_\alpha \tilde{C}, \quad E_\alpha = E + M_\alpha \tilde{F}$$

where

$$M_\alpha = -G_{\bullet\alpha}(N_{\alpha\bullet}G_{\bullet\alpha})^{-1}I_{\alpha\bullet},$$

then condition (iv) of Lemma 5.10 holds with C and F replaced by $\tilde{C}$ and $\tilde{F}$ respectively. Consequently, $\sum_{\alpha\in\mathcal{I}}\langle A_\alpha \mid \operatorname{im} E_\alpha \rangle$ is A_α-invariant for all α, and contains $\mathcal{A}$. Therefore, we can apply Corollary 5.11 to system (5.7). Before we do so, we first find a more compact form of $\sum_{\alpha\in\mathcal{I}}\langle A_\alpha \mid \operatorname{im} E_\alpha \rangle$.

Lemma 5.13 *Suppose that $R = 0$, $H = 0$, NG is a symmetric positive definite matrix and that $N(sI - A)^{-1}E$ is right invertible. Then we have*

$$\sum_{\alpha \in \mathcal{I}} \langle A_\alpha \mid \operatorname{im} E_\alpha \rangle = \langle A \mid \operatorname{im} \begin{bmatrix} E & G \end{bmatrix} \rangle.$$

Proof. From the discussion above and taking $\alpha = \varnothing$ in Lemma 5.10, we see that

$$\sum_{\alpha \in \mathcal{I}} \langle A_\alpha \mid \operatorname{im} E_\alpha \rangle = \langle A \mid \mathcal{A} + \mathcal{E} \rangle, \tag{5.22}$$

with $\mathcal{A}$ and $\mathcal{E}$ as in (5.12). Note that $\operatorname{im} E_\alpha \subseteq \operatorname{im} \begin{bmatrix} E & G \end{bmatrix}$ for each $\alpha \in \mathcal{I}$, so $\mathcal{E} \subseteq \operatorname{im} \begin{bmatrix} E & G \end{bmatrix}$. Furthermore, for every $\alpha, \beta \in \mathcal{I}$ we have $\operatorname{im}(A_\beta - A_\alpha) \subseteq \operatorname{im} G$, and hence $\mathcal{A} \subseteq \operatorname{im} G$. So we can conclude that $\mathcal{A} + \mathcal{E} \subseteq \operatorname{im} \begin{bmatrix} E & G \end{bmatrix}$.

To prove that the other inclusion also holds, choose $\alpha = \mathcal{I}_{n_\eta}$ and $\beta = \varnothing$, then we see that

$$\begin{bmatrix} A_\beta - A_\alpha & E_\beta - E_\alpha \end{bmatrix} = G(NG)^{-1} \begin{bmatrix} NA & NE \end{bmatrix}.$$

The right-invertibility of $NE + NA(sI - A)^{-1}E$ implies that $\begin{bmatrix} NA & NE \end{bmatrix}$ is of full row rank, and since NG is symmetric positive definite, this implies that

$$\operatorname{im} \begin{bmatrix} A_\beta - A_\alpha & E_\beta - E_\alpha \end{bmatrix} = \operatorname{im} G,$$

and hence $\operatorname{im} G \subseteq \mathcal{A} + \mathcal{E}$. By taking $\alpha = \varnothing$ we find that $\operatorname{im} E \subseteq \mathcal{E}$, and hence $\operatorname{im} \begin{bmatrix} E & G \end{bmatrix} \subseteq \mathcal{A} + \mathcal{E}$.

Together, this gives us

$$\mathcal{A} + \mathcal{E} = \operatorname{im} \begin{bmatrix} E & G \end{bmatrix},$$

which, combined with (5.22), proves the statement. ∎

Now, combining Corollary 5.11 with Lemma 5.12 and Lemma 5.13, we have the following result.

Theorem 5.14 *Suppose that $R = 0$, $H = 0$, NG is a symmetric positive definite matrix and that $N(sI - A)^{-1}E$ is right invertible as a rational matrix. Then the linear complementarity system (5.7) is disturbance decoupled if and only if*

$$\langle A \mid \operatorname{im} \begin{bmatrix} E & G \end{bmatrix} \rangle \subseteq \ker J.$$

Here we see that, although system (5.7) is highly non-linear and nonsmooth, the conditions for system (5.7) to be disturbance decoupled are geometric in nature and very akin to those for linear systems, for which $\langle A \mid \operatorname{im} E\rangle \subseteq \ker J$ is the condition. For the linear complementarity system we see that the effect of the complementarity variables on the state, captured by $\langle A \mid \operatorname{im} G\rangle$, also has to be taken into account.

5.5 CONCLUSIONS

In this chapter, we presented necessary and sufficient conditions, geometric in nature, under which a general linear multi-modal system is disturbance decoupled. The main results, presented in Theorem 5.8, Theorem 5.9, and Corollary 5.11 generalize almost all existing results in the literature on switched linear systems [Yurtseven et al., 2012], bimodal systems (Chapter 2), conewise linear systems (special case of Chapter 3), and linear complementarity systems of index zero (Chapter 4). In addition, these results led to necessary and sufficient conditions for a class of passive-like linear complementarity systems (see Theorem 5.14) whose disturbance decoupling properties have not been studied before.

For the presented general linear multi-modal system the necessary condition in Theorem 5.8 and the sufficient condition in Theorem 5.9 for being disturbance decoupled do not coincide. In Corollary 5.11 we presented several conditions under which these conditions do coincide.

In this chapter we only studied under what conditions a general linear multi-modal system is disturbance decoupled; rendering a system disturbance decoupled by means of feedback is the next step. Finding a static state feedback such that the resulting closed-loop system satisfies (5.9) becomes a linear algebraic problem and can be solved mimicking the footsteps for the linear case.

Possible future research lines include extending the results presented in this chapter to (discontinuous) piecewise affine systems and to study the extension to Filippov solutions. Furthermore, the results for the linear complementarity systems might be extended to the more general case with a not necessarily symmetric but positive semi-definite H for which there exists a positive symmetric matrix K such that $KGu = N^{\mathsf{T}}u$ for all $u \in \ker(H + H^{\mathsf{T}})$.

6 FAULT DETECTION AND ISOLATION FOR BIMODAL SYSTEMS

ABSTRACT: *We consider the problem of fault detection and identification for continuous bimodal piecewise-linear systems. Failures are modeled as disturbances from the nominal model and geometric techniques are employed to derive conditions under which one can detect and isolate faults. In addition, we discuss how an asymptotic observer can be designed by using the results of this chapter in combination with the existing results on observer design for bimodal systems. This chapter is based on the paper [Everts et al., 2016].*

6.1 INTRODUCTION

Fault detection and isolation (FDI) is an active area of research in control theory, due to the essential requirement of high reliability for many applications of control systems. Various types of FDI techniques have been proposed for linear systems and for some classes of nonlinear ones, see the comprehensive survey papers [Frank, 1990; Hwang et al., 2010; Isermann, 2006; Isermann and Bailé, 1997]. On the other hand, research on the FDI problem for hybrid and switched systems, and in particular for piecewise linear systems, has been less intensive and fruitful (see [Balluchi et al., 2002; Cocquempot et al., 2004; Narasimhan et al., 2000; Wang et al., 2009]).

In this chapter, we use the classical geometric control theory framework to investigate the problem of fault detection and isolation for bimodal piecewise linear systems. Our approach is inspired by the ideas pioneered in [Massoumnia, 1986a], where several formulations of the fault detection and isolation problem were stated and solved in geometric terms. As in [Massoumnia, 1986a], we consider the problem of how to define functions (called *residuals* in the following) of the system variables that are zero if no fault is present, and nonzero if a fault occurs. If the residual is nonzero, i.e. if a fault is affecting the system, the directional properties of such residuals give information on the type of failure. We give a sufficient condition for the residuals to provide sufficient information to solve the fault detection and isolation problem.

This chapter is organized as follows. We introduce the FDI problem in Section 6.2, by reviewing the FDI problem for linear systems. In Section 6.3 we consider the FDI problem for continuous piecewise-linear systems with two modes and state our main result in Theorem 6.8. We conclude the chapter in Section 6.4.

6.2 FAULT DETECTION AND ISOLATION

To introduce the problem of fault detection and isolation (FDI) we first review the formulation of the FDI problem for linear systems as treated in [Massoumnia, 1986a]. Consider the linear system described by

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^{k} F_i m_i(t) \tag{6.1a}$$

$$y(t) = Cx(t), \tag{6.1b}$$

with state $x(t) \in \mathbb{R}^{n_x}$, input $u(t) \in \mathbb{R}^{n_u}$, output $y(t) \in \mathbb{R}^{n_y}$, *fault modes* $m_i(t) \in \mathbb{R}^{f_i}$ for $i \in \mathcal{I}_k$, and matrices A, B, C and F_i of appropriate sizes. This system consists of a *nominal plant* described by (A, B, C) and by additional terms associated with the matrices F_i, the *fault signatures*. By choosing the F_is one can model different kind of faults. For example, by setting $F_1 = B_j$, where B_j is the j-th column of the input matrix B, the effect of a complete failure of the j-th actuator can be modeled by letting $m_1 = -u_j$. A biased actuator can be modeled by setting $m_1(\cdot)$ to be a constant function. Other types of faults (including changes in the system dynamics represented by the matrices A) can be accommodated in this framework too; see section III of [Massoumnia, 1986a] for more details. Without loss of generality we can assume that the F_is have full column rank and define

$$F := \begin{bmatrix} F_1 & F_2 & \cdots & F_k \end{bmatrix}.$$

In order to detect and identify faults, we define a Luenberger observer for the linear system (6.1)

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - K(y(t) - \hat{y}(t)) \tag{6.2a}$$

$$\hat{y}(t) = C\hat{x}(t) \tag{6.2b}$$

where the observer state $\hat{x}$ and output $\hat{y}$ are of the same dimensions as x and y respectively, and K is a to-be-designed $n_x \times n_y$-matrix. In addition, for this observer we define *residuals* r_i by

$$r_i(t) := D_i(y(t) - \hat{y}(t)) \qquad i \in \mathcal{I}_k, \tag{6.3}$$

where $D_1, D_2, \ldots, D_k$ are to-be-designed $n_y \times n_y$-matrices. In order to detect and identify faults, we want the ith residual to be nonzero exactly when the ith fault occurred.

Definition 6.1 The observer (6.2), (6.3) is a *fault detector* for system (6.1) if the following property holds whenever $x(0) = \hat{x}(0)$: $r_i(t) = 0$ for all $t \geqslant 0$ if and only if $m_i(t) = 0$ for almost all $t \geqslant 0$.

Remark 6.2 The assumption $x(0) = \hat{x}(0)$ corresponds to the observer being initialized with the correct value of the actual plant state. In practice, a stability requirement on the observer needs to be added to guarantee that the error $e(t) = x(t) - \hat{x}(t)$ converges to zero asymptotically. A discussion of such generic and more realistic case is given in Remark 6.11 below.

Clearly, a necessary condition for the existence of a fault detector for a linear system is that the fault matrix F should have full column rank, otherwise there would be no possibility of distinguishing faults that differ by elements in $\ker F$. Furthermore, to make sure that a nonzero m_i has an effect on the output y, the transfer matrix $C(sI - A)^{-1}F_i$ has to be left-invertible.

Before we state the main result for the FDI problem for linear systems, we define the following notions of separability and compatibility and present a proposition which connects these two notions. A collection of subspaces $\{\mathcal{T}_i\}_{i=1}^N$ is called *output separable* if

$$C\mathcal{T}_i \cap \left(\sum_{j \neq i} C\mathcal{T}_j \right) = \{0\} \tag{6.4}$$

for $i \in \mathcal{I}_N$ (see also formula (8) in [Massoumnia, 1986a]). We call a collection of (C, A)-invariant subspaces $\{\mathcal{T}_i\}_{i=1}^N$ *compatible* if they admit a common friend, i.e., there exists a map $K \in \mathbb{R}^{n_x \times n_y}$ such that

$$(A + KC)\mathcal{T}_i \subseteq \mathcal{T}_i, \tag{6.5}$$

for all $i \in \mathcal{I}_N$.

Proposition 6.3 ([Massoumnia, 1986a, Lemma 2]) *A set of output separable (C, A)-invariant subspaces $\{\mathcal{T}_i\}_{i=1}^N$ is compatible.*

We present the following variation on Theorem 3 in [Massoumnia, 1986a], using the notion of fault detectors.

Theorem 6.4 *If $C(sI-A)^{-1}F_i$ is left-invertible for $i \in \mathcal{I}_k$ and the subspaces $\{\mathcal{T}^*(F_i, C, A)\}_{i=1}^k$ are output separable, then the system (6.1) has a fault detector of the form (6.2), (6.3).*

Proof. Define $\mathcal{T}_i^* := \mathcal{T}^*(F_i, C, A)$, for $i \in \mathcal{I}_k$. From the assumption of output separability it follows from Proposition 6.3 that there is a map K that satisfies $(A+KC)\mathcal{T}_i^* \subseteq \mathcal{T}_i^*$ for all $i \in \mathcal{I}_k$. Furthermore, output separability guarantees that we can choose maps D_i such that $D_i y = y$ for $y \in C\mathcal{T}_i^*$ and $D_i y = 0$ for $y \in C\mathcal{T}_j^*$ whenever $j \neq i$, for $i, j \in \mathcal{I}_k$. We will show that with this choice of $D_1, D_2, \ldots, D_k$ and K the observer (6.2) is a fault detector for system (6.1).

Denote by $e := x - \hat{x}$ the error signal; its dynamics is given by

$$\dot{e}(t) = (A+KC)e(t) + \sum_{i=1}^{k} F_i m_i(t). \tag{6.6}$$

Since we assume that $e(0) = x(0) - \hat{x}(0) = 0$, the error $e(t)$ will stay in the subspace $\langle A+KC \mid \operatorname{im} F\rangle$ for all $t \geqslant 0$. For a given $i \in \mathcal{I}_k$, we write $e(t) = e_i(t) + \tilde{e}_i(t)$ where e_i and $\tilde{e}_i$ respectively satisfy

$$\dot{e}_i(t) = (A+KC)e_i(t) + F_i m_i(t) \tag{6.7}$$

$$\dot{\tilde{e}}_i(t) = (A+KC)\tilde{e}_i(t) + \sum_{j\neq i} F_j m_j(t) \tag{6.8}$$

with $e_i(0) = \tilde{e}_i(0) = 0$. By the choice of K it follows that $\mathcal{T}_j^* = \langle A+KC \mid \operatorname{im} F_j\rangle$ for all j, and consequently

$$\begin{aligned} \tilde{e}_i(t) \in \langle A+KC \mid \sum_{j\neq i} \operatorname{im} F_j\rangle &= \sum_{j\neq i} \langle A+KC \mid \operatorname{im} F_j\rangle \\ &= \sum_{j\neq i} \mathcal{T}_j^* \subseteq \ker D_i C \end{aligned}$$

for all t. Therefore, $r_i(t) = D_i C e_i(t) + D_i C \tilde{e}_i(t) = D_i C e_i(t)$ and r_i can thus be interpreted as an output of the linear system given by (6.7). Hence, if $m_i(t) = 0$ for almost all $t \geqslant 0$, then from (6.7) and $e_i(0) = 0$ we see that $e_i(t) = 0$ for all $t \geqslant 0$, and as a result $r_i(t) = D_i C e_i(t) = 0$ for all $t \geqslant 0$.

On the other hand, by using the identity

$$(sI-X)^{-1} - (sI-Y)^{-1} = (sI-X)(X-Y)(sI-Y)^{-1},$$

it can be verified that

$$C(sI - A - KC)^{-1}F_i = \\ (I - C(sI - A - KC)^{-1}K)C(sI - A)^{-1}F_i.$$

Since $C(sI - A)^{-1}F_i$ and $I - C(sI - A - KC)^{-1}K$ are both left-invertible, so is $C(sI - A - KC)^{-1}F_i$. Because $D_i y = y$ for all $y \in C\mathcal{T}_i^* = C\langle A + KC \mid \operatorname{im} F_i\rangle$, we have that $D_i C(sI - A - KC)^{-1}F_i = C(sI - A - KC)^{-1}F_i$. Therefore, the transfer matrix from m_i to r_i is left-invertible. Consequently, if $r_i(t) = 0$ for all $t \geqslant 0$, then $m_i(t) = 0$ for almost all $t \geqslant 0$.

Hence, this choice of $D_1, D_2, \ldots, D_k$ and K results in a fault detector for system (6.1). ■

6.3 FAULT DETECTION AND ISOLATION FOR BIMODAL SYSTEMS

We now consider the FDI problem for continuous piecewise-linear systems with two modes. Such systems are described by

$$\dot{x}(t) = \begin{cases} A_1 x + Bu + \sum_{i=1}^{k} F_i m_i & \text{if } c^\top x \leqslant 0 \\ A_2 x + Bu + \sum_{i=1}^{k} F_i m_i & \text{if } c^\top x \geqslant 0 \end{cases} \tag{6.9a}$$

$$y = Cx, \tag{6.9b}$$

where $A_j \in \mathbb{R}^{n_x \times n_x}$ for $j = 1, 2$, $c \in \mathbb{R}^{n_x}$, and x, y, u, m_i, B, C and F_i are as before. We assume that the right-hand side of (6.9a) is continuous in x, which is equivalent to saying that A_1 and A_2 satisfy

$$A_1 - A_2 = hc^\top \tag{6.10}$$

for some $h \in \mathbb{R}^{n_x}$. Without loss of generality we can assume that every fault signature F_i has full column rank and we define

$$\begin{aligned} F &:= \begin{bmatrix} F_1 & F_2 & \cdots & F_k \end{bmatrix} \\ m &:= \begin{bmatrix} m_1^\top & m_2^\top & \cdots & m_k^\top \end{bmatrix}^\top. \end{aligned}$$

Similar to the linear case, we aim at designing a Luenberger-type observer that produces a residual signal which provides information about the presence and the nature of failures on the basis of the inputs and outputs of the plant. The dynamics of the to-be-designed observer are

$$\dot{\widehat{x}} = \begin{cases} A_1\widehat{x} + Bu - K(y - \widehat{y}) & \text{if } c^\mathsf{T}\widehat{x} \leqslant 0 \\ A_2\widehat{x} + Bu - K(y - \widehat{y}) & \text{if } c^\mathsf{T}\widehat{x} \geqslant 0 \end{cases} \tag{6.11a}$$

$$\widehat{y} = C\widehat{x} \tag{6.11b}$$

$$r_i = D_i(y - \hat{y}), \qquad \forall i \in \mathcal{I}_k, \tag{6.11c}$$

where $\widehat{x} \in \mathbb{R}^{n_x}$, $\widehat{y} \in \mathbb{R}^{n_y}$, r_i is the ith residual, and $K \in \mathbb{R}^{n_x \times n_y}$ and $D_i \in \mathbb{R}^{n_y \times n_y}$ are design parameters. Analogously to the linear case, we call observer (6.11) a *fault detector* for system (6.9) if the following holds whenever $x(0) = \hat{x}(0)$: $r_i(t) = 0$ for all $t \geqslant 0$ if and only if $m_i(t) = 0$ for almost all $t \geqslant 0$.

To study the dynamics of the residuals, we define the error by $e(t) := x(t) - \hat{x}(t)$. The dynamics of e depends on both the mode of the system (6.9) and the mode of the observer (6.11), and hence has four modes, described by:

$$\dot{e} = \begin{cases} (A_1 + KC)e + Fm & \text{if } c^\mathsf{T}x \leqslant 0, c^\mathsf{T}\widehat{x} \leqslant 0 \\ (A_1 + KC)e + hc^\mathsf{T}\widehat{x} + Fm & \text{if } c^\mathsf{T}x \leqslant 0, c^\mathsf{T}\widehat{x} \geqslant 0 \\ (A_2 + KC)e - hc^\mathsf{T}\widehat{x} + Fm & \text{if } c^\mathsf{T}x \geqslant 0, c^\mathsf{T}\widehat{x} \leqslant 0 \\ (A_2 + KC)e + Fm & \text{if } c^\mathsf{T}x \geqslant 0, c^\mathsf{T}\widehat{x} \geqslant 0. \end{cases} \tag{6.12}$$

We now define some subspaces that play a similar role as the subspaces $\mathcal{T}^*(F_i, C, A)$ for the linear system FDI situation:

$$\mathcal{Z}_0^j \;:=\; \mathcal{T}^*(A_j, C, \operatorname{im} h) \tag{6.13a}$$

$$\mathcal{Z}_i^j \;:=\; \mathcal{T}^*(A_j, C, \operatorname{im} F_i) \tag{6.13b}$$

$$\mathcal{W}_i^j \;:=\; \mathcal{T}^*(A_j, C, \operatorname{im} \begin{bmatrix} h & F_i \end{bmatrix}) \tag{6.13c}$$

for $j = 1, 2$ and $i \in \mathcal{I}_k$. In these subspaces, we treat h analogously to a fault signature; this will enable us to distinguish between mode-switching and faults. Firstly, we show that some of these subspaces coincide.

Proposition 6.5 *Let* $\mathcal{Z}_i^j$ *and* $\mathcal{W}_i^j$ *be as in* (6.13a)-(6.13c)*. Then* $\mathcal{Z}_0^1 = \mathcal{Z}_0^2$ *and* $\mathcal{W}_i^1 = \mathcal{W}_i^2$ *for* $i \in \mathcal{I}_k$*. Moreover,* $\mathcal{Z}_0^j \subseteq \mathcal{W}_i^j$ *and* $\mathcal{Z}_i^j \subseteq \mathcal{W}_i^j$ *for* $j = 1, 2$ *and* $i \in \mathcal{I}_k$*.*

Proof. Since $\mathcal{Z}_0^1$ is (C, A_1)-invariant, there exists a map $K \in \mathbb{R}^{n\times p}$ such that $(A_1 + KC)\mathcal{Z}_0^1 \subseteq \mathcal{Z}_0^1$. From equation (6.10) it follows that for any $z \in \mathcal{Z}_0^1$ we have

$$\begin{aligned}(A_2 + KC)z &= (A_1 + KC - hc^{\mathsf{T}})z \\ &= (A_1 + KC)z - hc^{\mathsf{T}}z \\ &\in \mathcal{Z}_0^1,\end{aligned}$$

since $\mathcal{Z}_0^1$ is invariant under $A_1 + KC$ and contains h. It follows that $\mathcal{Z}_0^1$ is also (C, A_2)-invariant, and since $\mathcal{Z}_0^2$ is the smallest (C, A_2)-invariant subspace containing im h, we conclude that $\mathcal{Z}_0^1 \subseteq \mathcal{Z}_0^2$. A symmetric argument proves the converse inclusion, and hence we conclude that $\mathcal{Z}_0^1 = \mathcal{Z}_0^2$. The equality $\mathcal{W}_i^1 = \mathcal{W}_i^2$ can be proved in an analogous manner.

The inclusions $\mathcal{Z}_0^j \subseteq \mathcal{W}_i^j$ and $\mathcal{Z}_i^j \subseteq \mathcal{W}_i^j$, for $j = 1, 2$ and $i \in \mathcal{I}_k$, follow from the definition of the subspaces, and from the fact that $\operatorname{im} h$ and $\operatorname{im} F_i$ are both contained in $\operatorname{im} \begin{bmatrix} h & F_i \end{bmatrix}$. ■

In light of the result of Proposition 6.5, in the following we denote the subspaces $\mathcal{W}_i^1 = \mathcal{W}_i^2$ simply by $\mathcal{W}_i$ for $i \in \mathcal{I}_k$, and the subspace $\mathcal{Z}_0^1 = \mathcal{Z}_0^2$ by $\mathcal{Z}_0$. Next, we show that under the assumption $\ker C \subseteq \ker c^{\mathsf{T}}$, the subspaces $\mathcal{Z}_i^1$ and $\mathcal{Z}_i^2$ also coincide.

Proposition 6.6 *If* $\ker C \subseteq \ker c^{\mathsf{T}}$*, then* $\mathcal{Z}_i^1 = \mathcal{Z}_i^2$ *for* $i \in \mathcal{I}_k$.

Proof. We first prove that $\mathcal{Z}_i^1$ is (C, A_2)-invariant. Since $\ker C \subseteq \ker c^{\mathsf{T}}$, we have $hc^{\mathsf{T}}|_{\ker C} = \{0\}$. Hence, using $A_2 = A_1 - hc^{\mathsf{T}}$ we see that

$$\begin{aligned}A_2(\mathcal{Z}_i^1 \cap \ker C) &= (A_1 - hc^{\mathsf{T}})(\mathcal{Z}_i^1 \cap \ker C) \\ &= A_1(\mathcal{Z}_i^1 \cap \ker C) \\ &\subseteq \mathcal{Z}_i^1,\end{aligned}$$

where the last inclusion follows from the fact that $\mathcal{Z}_i^1$ is (C, A_1)-invariant. Consequently, $\mathcal{Z}_i^1$ is (C, A_2)-invariant; moreover, it contains $\operatorname{im} F_i$. Since $\mathcal{Z}_i^2$ is the smallest such subspace, it follows that $\mathcal{Z}_i^2 \subseteq \mathcal{Z}_i^1$. An analogous argument proves that $\mathcal{Z}_i^2$ is also (C, A_1)-invariant, and consequently $\mathcal{Z}_i^1 \subseteq \mathcal{Z}_i^2$. The claimed equality follows. ■

When under the assumption of Prop. 6.6, the subspace $\mathcal{Z}_i^1 = \mathcal{Z}_i^2$ will be denoted by $\mathcal{Z}_i$. The output separability of the subspaces $\{\mathcal{Z}_i^1\}_{i=0}^k$ (or $\{\mathcal{Z}_i^2\}_{i=0}^k$) has some important consequences.

Proposition 6.7 *If $\{\mathcal{Z}_i^1\}_{i=0}^k$ and $\{\mathcal{Z}_i^2\}_{i=0}^k$ are output separable, then $\mathcal{W}_i = \mathcal{Z}_0^j + \mathcal{Z}_i^j$, for $j = 1,2$ and $i \in \mathcal{I}_k$. Moreover, the subspaces $\{\mathcal{W}_i\}_{i=1}^k$ are compatible.*

Proof. We prove the claim for $j = 1$, the case $j = 2$ being completely analogous. From the inclusions $\mathcal{Z}_0^1 \subseteq \mathcal{W}_i^1$ and $\mathcal{Z}_i^1 \subseteq \mathcal{W}_i^1$, $i \in \mathcal{I}_k$ proved in Proposition 6.5 conclude that $\mathcal{Z}_0^1 + \mathcal{Z}_i^1 \subseteq \mathcal{W}_i$ for $i \in \mathcal{I}_k$.

Using Proposition 6.3 we conclude that $\{\mathcal{Z}_i^1\}_{i=0}^k$ have a common friend K. Since $(A_1 + KC)\mathcal{Z}_i^1 \subseteq \mathcal{Z}_i^1$, $i = 0,1,\dots,k$, it follows that $(A_1 + KC)(\mathcal{Z}_0^1 + \mathcal{Z}_i^1) \subseteq (\mathcal{Z}_0^1 + \mathcal{Z}_i^1)$. Consequently $\mathcal{Z}_0^1 + \mathcal{Z}_i^1$ is a (C, A_1)-invariant subspace for $i \in \mathcal{I}_k$. Note that $\mathcal{Z}_0^1 + \mathcal{Z}_i^1$ contains $\operatorname{im} h$ and $\operatorname{im} F_i$, and that $\mathcal{W}_i$ is the smallest (C, A_1)-conditioned invariant with this property; it follows that $\mathcal{Z}_0^1 + \mathcal{Z}_i^1 \supseteq \mathcal{W}_i$, and hence $\mathcal{W}_i = \mathcal{Z}_0^1 + \mathcal{Z}_i^1$ for $i \in \mathcal{I}_k$. Since K is a common friend of the subspaces $\{\mathcal{W}_i\}_{i=1}^k$, the subspaces $\{\mathcal{W}_i\}_{i=1}^k$ are compatible. ∎

In the proof of the following theorem, which is the main result of this chapter, we treat state-switching analogously to a fault occurrence, with fault signature h.

Theorem 6.8 *Consider system (6.9), and assume* $\ker C \subseteq \ker c^\top$. *If $C(sI - A)^{-1}F_i$ is left-invertible for all $1 \leqslant i \leqslant k$ and the subspaces $\{\mathcal{Z}_i\}_{i=0}^k$ in (6.13a)-(6.13b) are output separable, then there exists a fault detector of the form (6.11) for system (6.9).*

Proof. By Proposition 6.3 the subspaces $\{\mathcal{Z}_i\}_{i=0}^k$ are compatible; let K be a common friend for these subspaces. For each $i \in \mathcal{I}_k$ let the map D_i be such that $D_i y = y$ for $y \in C\mathcal{Z}_i$ and

$$C\mathcal{Z}_j \subseteq \ker D_i \tag{6.14}$$

for $j = 0,1,\dots,k$ with $j \neq i$. We show that with this choice of $D_1, D_2, \dots, D_k$ and K, the observer (6.11) is a fault detector for system (6.9).

Given a initial state x_0, an input $u(\cdot)$ and a fault signal $m(\cdot)$, the trajectories of x and $\hat{x}$ are known and can be used to the define the following function:

$$\theta(t) := \begin{cases} 0 & \text{if } c^\top x(t) \leqslant 0,\ c^\top \hat{x}(t) \leqslant 0 \\ c^\top \hat{x}(t) & \text{if } c^\top x(t) \leqslant 0,\ c^\top \hat{x}(t) \geqslant 0 \\ c^\top x(t) & \text{if } c^\top x(t) \geqslant 0,\ c^\top \hat{x}(t) \leqslant 0 \\ c^\top (\hat{x}(t) - x(t)) & \text{if } c^\top x(t) \geqslant 0,\ c^\top \hat{x}(t) \geqslant 0. \end{cases}$$

Using $\theta(t)$, we can rewrite the error dynamics (6.12) as

$$\dot{e}(t) = (A_1 + KC)e(t) + h\theta(t) + Fm(t). \tag{6.15}$$

Hence, the error e can be seen as a solution of a linear system with inputs θ and m. Furthermore, for a given index $i \in \mathcal{I}_k$ we can write the error e as $e = e_i + \tilde{e}_i$, where e_i is the part of the error that corresponds to the effects of fault m_i, with e_i and $\tilde{e}_i$ satisfying

$$\dot{e}_i(t) = (A_1 + KC)e_i(t) + F_i m_i(t) \tag{6.16}$$

$$\dot{\tilde{e}}_i(t) = (A_1 + KC)\tilde{e}_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{k} F_j m_j(t) + h\theta(t) \tag{6.17}$$

and $e_i(0) = \tilde{e}_i(0) = 0$. We write $F_0 = h$ and note that from the choice of K it follows that

$$\begin{aligned} \tilde{e}_i(t) &\in \langle A + KC \mid \sum_{\substack{j=0 \\ j \neq i}}^{k} \operatorname{im} F_j \rangle \\ &= \sum_{\substack{j=0 \\ j \neq i}}^{k} \langle A + KC \mid \operatorname{im} F_j \rangle = \sum_{\substack{j=0 \\ j \neq i}}^{k} \mathcal{Z}_j \\ &\subseteq \ker D_i C, \end{aligned}$$

for all t. Therefore, $r_i(t) = D_i C e_i(t) + D_i C \tilde{e}_i(t) = D_i C e_i(t)$ and r_i can thus be seen as an output of the linear system described by (6.16).

Suppose that $m_i(t) = 0$ for almost all $t \geqslant 0$. Then from (6.16) and $e_i(0) = 0$ we see that $e_i(t) = 0$ for all $t \geqslant 0$, and hence $r_i(t) = D_i C e_i(t) = 0$.

On the other hand, by using the identity

$$(sI - X)^{-1} - (sI - Y)^{-1} = (sI - X)(X - Y)(sI - Y)^{-1}$$

for two square matrices X and Y, we conclude that

$$\begin{aligned} &C(sI - A_1 - KC)^{-1}F_i = \\ &\qquad (I - C(sI - A_1 - KC)^{-1}K)C(sI - A_1)^{-1}F_i. \end{aligned}$$

Since both $I - C(sI - A_1 - KC)^{-1}K$ and $C(sI - A_1)^{-1}F_i$ are left-invertible as rational matrices, so is $C(sI - A_1 - KC)^{-1}F_i$. The choice of D_i guarantees that the transfer function $D_i C(sI -$

$A_1 - KC)^{-1}F_i$ equals $C(sI - A_1 - KC)^{-1}F_i$ and is therefore left-invertible as well. Consequently, if $r_i(t) = 0$ for all $t \geqslant 0$, then $m_i(t) = 0$ for almost all $t \geqslant 0$. ■

Remark 6.9 The assumption that $\ker C \subseteq \ker c^\top$ implies that $c^\top = h^\top C$ for some vector h. Consequently, the mode that the system is in at time t can be determined based on the output $y(t)$.

Remark 6.10 If all the fault signals are scalar, i.e. $f_i = 1$ for all i, then all the F_is are vectors. In this case left-invertibility of $C(sI - A_1 - KC)F_i$ is equivalent with input observability of the system given by (6.16) with output Ce_i. Furthermore, if

$$\{0\} = \operatorname{im} F_i \cap \ker C,$$

for $i \in \mathcal{I}_k$, then using Algorithm 4.1.1 p. 202 of [Basile and Marro, 1992] we conclude that $\mathcal{Z}_i^j = \operatorname{im} F_i$, $i \in \mathcal{I}_k$, $j = 1, 2$, in which case the assumption $\ker C \subseteq \ker c^\top$ is not needed in Theorem 6.8 since $\mathcal{Z}_i^1 = \mathcal{Z}_i^2$ is already satisfied for all $i = 0, 1, \ldots, k$.

Remark 6.11 In the generic case $e(0) \neq 0$ the issue of the asymptotic stability of the observer comes into play: the gain K must not only be a common friend of the $\mathcal{Z}_i$ but also guarantee that $e(t) \to 0$ as $t \to 0$ in the absence of faults. Then, if (the norm of) $r_i(t)$ is above a certain threshold, one can observe that a fault has occurred. The following sufficient condition for the existence of asymptotic observers for bimodal systems has been proved in [Juloski et al., 2007].

Proposition 6.12 *Consider the plant (6.9) with $m(\cdot) = 0$, and the observer (6.11). If there exist matrices $P = P^\top > 0$, K, M, and $\lambda, \mu \in \mathbb{R}$ with $\lambda \geqslant 0$, $\mu > 0$ such that*

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \leqslant 0, \tag{6.18}$$

where

$$\begin{aligned} X_{11} &:= (A_1 + KC)^\top P + P(A_1 + KC) + \mu I_n \\ X_{12} &:= -Phc^\top + \frac{\lambda}{2}(c - C^\top M)c^\top \\ X_{21} &:= X_{12}^\top \\ X_{22} &:= -\lambda c c^\top, \end{aligned}$$

then the error dynamics (6.12) is globally asymptotically stable in the sense of Lyapunov.

To apply Proposition 6.12 in the setting for FDI described in the previous section, the stabilizing observer gain K solving the matrix inequality (6.18) must also be a common friend of the subspaces $\{\mathcal{Z}_i\}_{i=0}^{k}$. Replacing K in (6.18) by a parametrization of the common friends of the $\mathcal{Z}_i$, one finds a *bilinear* matrix inequality since it contains products of the indeterminates. This results in a non-convex optimization problem. However, such bilinear matrix inequalities can be sub-optimally solved (i.e. local optima can be computed) for instance by employing standard LMI solvers, e.g. Yalmip (see [Löfberg, 2004]).

6.4 CONCLUSIONS

We have illustrated a geometric approach to the design of fault detection systems for a class of bimodal piecewise-linear systems. Sufficient conditions (Theorem 6.8) have been given for the existence of an observer that produces residuals that are sufficiently informative about the fault. A method for finding an asymptotic observer based on bilinear matrix inequalities has been discussed. Extension of the ideas and results towards multi-modal piecewise linear systems is a possible research direction.

In the next chapter we continue with the FDI problem for a class of linear dynamical systems defined over an undirected graph.

7 FAULT DETECTION AND ISOLATION FOR SYSTEMS DEFINED OVER GRAPHS

ABSTRACT: *In this chapter we consider the problem of fault detection and isolation for a class of linear dynamical systems defined over a graph containing faultable vertices and observer vertices. Using a geometric approach, we provide a characterization of the smallest conditioned invariant subspaces generated by faults in terms of the underlying graph structure. Based on this characterization, we give graph-theoretic conditions guaranteeing fault detectability. In addition, we provide a condition under which fault detectability fails. This chapter is based on our conference paper [Rapisarda et al., 2015].*

7.1 INTRODUCTION

Detecting and identifying faults in multi-agent systems is particularly relevant, as faults either in the agents or in the communication structure can have serious consequences. This leads naturally to considering design issues. Of primary importance is designing a communication structure that guarantees the prompt detection and accurate identification of faults.

In the literature on multi-agent systems, within the control community several approaches have been proposed for fault detection and isolation. Among these, those closer to the results presented here are [Jafari et al., 2011; Pasqualetti et al., 2012; Rahimian and Aghdam, 2013]. In these references the geometric approach to fault detection and isolation based on unknown input observers pioneered in [Massoumnia, 1986b,a] is used. In [Jafari et al., 2011; Rahimian and Aghdam, 2013] problems of leader collocation to guarantee controllability of a multi-agent system under communication failure are studied; in [Pasqualetti et al., 2012] the authors study the problem of reliable computation in consensus networks.

In this chapter, we consider a class of linear dynamical systems defined over an undirected graph. In this network, we identify two disjoint sets of agents: the faultable agents, which are prone to failure, and the observer agents, whose output is measurable. Faults such as total communication failures, biased sensing, etc. can be modeled in a straightforward way in our framework. Fault detection is performed by an unknown input observer, and stated in the geometric language of [Massoumnia, 1986a], i.e. output separability of fault subspaces.

First we present a characterization of the smallest conditioned invariant subspaces that are generated by the faults. This characterization is exploited in order to give graph-theoretical conditions guaranteeing output separability in terms of distances between faultable agents and observer ones. In addition, we study the case where two faultable vertices share exactly the same neighbors in order to present a condition under which fault detectability fails.

The organization of this chapter is as follows. Section 7.2 introduces graph-theoretical tools that will be used later on in this chapter. Section 7.3 discusses the problem of fault detection in a geometric setting. In section 7.4, we present the main results of this chapter. Finally, the chapter closes with the conclusions in section 7.5.

7.2 BACKGROUND MATERIAL

Consider a *simple* graph $G = (V, E)$, i.e. *undirected* and *unweighted* graph containing no multiple edges or loops on vertices. Let the vertex set V be given by

$$V = \{1, 2, \ldots, n\}$$

and the edge set E be a subset of all unordered pairs of vertices, that is, $E \subseteq \{\{i, j\} \mid i, j \in V\}$.

Associated to such a graph $G = (V, E)$, we define a family of matrices, called the *qualitative class* of G, by

$$\mathcal{Q}(G) = \{X \in \mathbb{R}^{n \times n} \mid \text{ for } i \neq j,\ X_{ij} \neq 0 \Leftrightarrow \{i, j\} \in E\}.$$

Typical elements of the qualitative class are adjacency, Laplacian and normalized Laplacian matrices corresponding to a simple graph. Note that the elements of the qualitative class may correspond to matrices associated to a weighted graph, even though the graph G is itself simple and hence unweighted.

We say that a graph $H = (V, E)$ is *bipartite* if there exist disjoint vertex sets V^- and V^+ such that $V^- \cup V^+ = V$ and the edge set E contains only edges connecting one vertex from V^- and the other from V^+. With a slight abuse of notation, we write $H = (V^-, V^+, E)$ for bipartite graphs.

Associated to a bipartite graph $H = (V^-, V^+, E)$ with

$$V^- = \{r_1, r_2, \ldots, r_p\}$$
$$V^+ = \{c_1, c_2, \ldots, c_q\},$$

we define a *pattern class* by

$$\mathcal{P}(H) := \{Y \in \mathbb{R}^{p\times q} \mid Y_{ij} \neq 0 \Leftrightarrow \{r_i, c_j\} \in E\}.$$

Note that the matrices in the pattern class are not necessarily square unlike those in the qualitative class.

A set of t edges of a bipartite graph $H = (V^-, V^+, E)$ that do not share a common vertex is called a *t-matching*. A t-matching is called *constrained* if there is no other t-matching between the matched vertices.

The following is a classical result (see [Hershkowitz and Schneider, 1993, Thm. 3.9]) in the study of the minimal rank of a given pattern class.

Theorem 7.1 *Let $H = (V^-, V^+, E)$ be a bipartite graph with vertex sets $V^- = \{r_1, r_2, \ldots, r_p\}$ and $V^+ = \{c_1, c_2, \ldots, c_q\}$. All matrices $Y \in \mathcal{P}(H)$ are of full rank if and only if H admits a constrained* $\min(p, q)$*-matching.*

7.3 PROBLEM STATEMENT

Let $G = (V, E)$ be a *simple* and connected graph with the vertex set

$$V = \{1, 2, \ldots, n\}.$$

Two subsets of V will play an important role in the sequel. We denote these sets by V_F (*faultable vertices*) and V_O (*observer vertices*). For simplicity, we assume that these sets are disjoint and that the first q vertices are faultable and the last s are observer, that is

$$\begin{aligned} V_F &= \{1, 2, \ldots, q\} \\ V_O &= \{n-s+1, n-s+2, \ldots, n\}. \end{aligned}$$

Throughout this chapter, we will consider systems of the form

$$\dot{x}(t) = Xx(t) + Mf(t) \tag{7.1a}$$

$$y(t) = Nx(t) \tag{7.1b}$$

where x is the state, f is the fault mode and y is the output vector. The matrices X, M and N are related to the given simple graph G and the pair (V_F, V_O) in the sense that X belongs to

$\mathcal{Q}(G)$, M encodes the faultable vertices and N the observer vertices, that is

$$M = \begin{bmatrix} I_q \\ 0_{n-q,q} \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0_{s,n-s} & I_s \end{bmatrix}.$$

The problem we will address in this chapter amounts to setting up an observer

$$\dot{\widehat{x}}(t) = (X + KN)\widehat{x}(t) - Ky(t) \tag{7.2a}$$
$$\widehat{y}(t) = N\widehat{x}(t) \tag{7.2b}$$

where $K \in \mathbb{R}^{n\times s}$, in order to detect if and which faults are active.

To elaborate on what we mean by detecting faults, define the error by

$$e(t) := \widehat{x}(t) - x(t)$$

and note that it satisfies the dynamics

$$\dot{e}(t) = (X + KN)e(t) - Mf(t) \tag{7.3a}$$
$$r(t) = Ne(t) \tag{7.3b}$$

where r is the *residual* term.

Assuming that $e(0) = 0$, if only the i-th fault occurs, i.e., if $f_i \neq 0$ and $f_j = 0$ for $j \neq i$, the error in (7.3) is confined to the smallest $(X + KN)$-invariant subspace containing $\operatorname{im} M_i$, where M_i denotes the ith column of the matrix M. Under such strong assumption (implying that the observer is initialized precisely with the same initial conditions as the system), the fault detection and isolation problem can be stated in the geometric language illustrated in sect. 7.2 as follows:

> *Given the dynamics (7.1), find a family $\{\mathcal{T}_i\}_{i=1}^{q}$ of subspaces of $\mathbb{R}^n$ and associated $K \in \mathbb{R}^{n\times s}$ such that*
>
> *C1)* $(X + KN)\mathcal{T}_i \subseteq \mathcal{T}_i,\ i \in \mathcal{I}_q$,
>
> *C2)* $\operatorname{im} M_i \subseteq \mathcal{T}_i,\ i \in \mathcal{I}_q$,
>
> *C3)* $N\mathcal{T}_i \cap \left(\sum_{j\neq i} N\mathcal{T}_j\right) = \{0\},\ i \in \mathcal{I}_q$.

If such a family of subspaces exists, $e(0) = 0$, and only the i-th fault occurs, then it follows from the structure of the observer (7.2) that $e(t)$ belongs to $\mathcal{T}_i$ for all $t \in \mathbb{R}$ (condition C1 and

C2). Because of output separability (condition C3, see (6.4)), in case of multiple faults, information on the presence and type of faults can be obtained by projecting the residual vector r on $N\mathcal{T}_i$ for $i \in \mathcal{I}_q$.

In the generic case of $e(0) \neq 0$, the asymptotic stability of the observer comes into play, and the common friend K (condition C1, see (6.5)) of the family $\{\mathcal{T}_i\}$ must be such that

$$\lim_{t\to\infty} e(t) = 0,$$

i.e. the conditions C1–C3 must be supplemented by the condition

$$\boxed{\textit{C4)}\ \ \sigma(X + KN) \subset \mathbb{C}_-.}$$

The main goal of this chapter is to find graph-theoretical sufficient conditions that enable fault detection and isolation as explained above.

7.4 GRAPH-THEORETIC CONDITIONS

Given particular choices of X, M and N in (7.1), checking for the existence of an output separable family $\{\mathcal{T}_i\}_{i=1}^q$ that satisfies conditions C1–C3 is a straightforward linear algebra problem. For this, we define $\mathcal{T}_i^*$ to be $\mathcal{T}^*(M_i, N, X)$; the smallest (N, X)-invariant subspace containing $\operatorname{im} M_i$ for $i \in \mathcal{I}_q$. Then the following result holds.

Lemma 7.2 *Consider the system* (7.1). *There exists a family of subspaces* $\{\mathcal{T}_i\}_{i=1}^q$ *satisfying conditions C1–C3 if and only if* $\{\mathcal{T}_i^*\}_{i=1}^q$ *is output separable.*

Proof. See the proof of Theorem 3 p. 841 of [Massoumnia, 1986a]. ∎

Thus in order to check whether the FDI approach of section 7.3 can be applied to a given network, one needs to compute the (N, X)-invariant subspaces $\mathcal{T}_i^*$ via the algorithm (1.8), and then check the output separability of $\{\mathcal{T}_i^*\}_{i=1}^q$. Necessary and sufficient conditions for the existence of a stabilizing common friend of subspaces $\mathcal{T}_i$ are given in Theorems 9 and 10 of [Massoumnia, 1986a].

In this section we pursue another line of thought, providing a sufficient condition for output separability *based on graph-theoretical considerations*, as in [Jafari et al., 2011; Rahimian and Aghdam, 2013]. Such approach has the advantage of avoiding potential numerical and computational complexity issues associated with linear algebra computations for large-scale networks, providing instead robust conditions based on discrete mathematics. Moreover, it offers insight into the structural properties of networks, with potentially useful applications for example in the design of network systems.

Moreover, the conditions we will provide are going to be valid not only for a particular choice of the matrix X but rather for a family of matrices within the qualitative class, namely the so-called *distance-information preserving* matrices.

To elaborate on this class of matrices, recall that for a graph $G = (V, E)$ the *distance* between two vertices is the length of the shortest path connecting them. The distance between the vertices i and j is denoted by $\operatorname{dist}(i, j)$. By convention, $\operatorname{dist}(i, j) := \infty$ if no path exists between vertex i and vertex j, and $\operatorname{dist}(i, i) = 0$ for any vertex i. In this chapter we assume that the graph G is connected, hence $\operatorname{dist}(i, j) < \infty$ for all vertices i and j.

Definition 7.3 A matrix $X \in \mathbb{R}^{n\times n}$ is *distance-information preserving* with respect to the graph $G = (V, E)$ if

$$(X^k)_{i,j} \begin{cases} = 0 & \text{if } \operatorname{dist}(i,j) > k, \\ \neq 0 & \text{if } \operatorname{dist}(i,j) = k \end{cases}$$

for $k \geqslant 0$.

Clearly, every distance-information preserving matrix belongs to the qualitative class $\mathcal{Q}(G)$ but the converse is not true in general. Laplacian and adjacency matrices, frequently used in describing graph structures, are typical instances of distance-information preserving matrices.

Furthermore, we define the distance of a vertex $i \in V$ from a nonempty subset of vertices $V' \subseteq V$ as follows:

$$\operatorname{dist}(i, V') := \min_{j\in V'} \operatorname{dist}(i, j).$$

We begin with presenting a characterization of the subspaces $\mathcal{T}_i^*$.

Lemma 7.4 *Consider the system (7.1). Suppose that X is a distance-information preserving matrix with respect to the simple and connected graph $G = (V, E)$. Let $i \in V_F$ and $d_i = \operatorname{dist}(i, V_O)$. Then, we have*

$$\mathcal{T}_i^* = \operatorname{im} \begin{bmatrix} M_i & X_i & \cdots & (X^{d_i})_i \end{bmatrix} \tag{7.4a}$$

$$N\mathcal{T}_i^* = \operatorname{im} N(X^{d_i})_i \tag{7.4b}$$

where X_i denotes the ith column of the matrix X.

Proof. We prove the statement by employing the recursion (1.8), starting with

$$\mathcal{T}^0 = \operatorname{im} M_i.$$

Since $V_F \cap V_O = \varnothing$, it follows that

$$\operatorname{im} M_i \subseteq \ker N. \tag{7.5}$$

Thus, we obtain

$$\begin{aligned} \mathcal{T}^1 &= \operatorname{im} M_i + X\left(\mathcal{T}^0 \cap \ker N\right) \\ &= \operatorname{im} M_i + X\left(\operatorname{im} M_i \cap \ker N\right) \\ &= \operatorname{im} M_i + X \operatorname{im} M_i. \end{aligned}$$

Note that $X \operatorname{im} M_i = \operatorname{im} X_i$ by the definition of M. Hence, we get

$$\mathcal{T}^1 = \operatorname{im} M_i + \operatorname{im} X_i.$$

Since $V_F \cap V_O = \varnothing$, we have $d_i > 0$. Together with the connectedness of G this implies that $\mathcal{T}^1$ is strictly larger than $\mathcal{T}^0$.

Next, we note that

$$\begin{aligned} \mathcal{T}^2 &= \operatorname{im} M_i + X\left(\mathcal{T}^1 \cap \ker N\right) \\ &= \operatorname{im} M_i + X\big((\operatorname{im} M_i + \operatorname{im} X_i) \cap \ker N\big). \end{aligned}$$

It follows from (7.5) that

$$(\operatorname{im} M_i + \operatorname{im} X_i) \cap \ker N = \operatorname{im} M_i + (\operatorname{im} X_i \cap \ker N).$$

Now, we distinguish the following two cases: $\operatorname{dist}(i, V_O) = 1$ and $\operatorname{dist}(i, V_O) > 1$. In the first case, we have

$$\operatorname{im} X_i \cap \ker N = \{0\}$$

and hence we find

$$\mathcal{T}^2 = \operatorname{im} M_i + \operatorname{im} X_i = \mathcal{T}^1,$$

and the algorithm stops. In the second case we have

$$\operatorname{im} X_i \subseteq \ker N, \tag{7.6}$$

which further implies that

$$\operatorname{im} X_i \cap \ker N = \operatorname{im} X_i.$$

Therefore, we obtain

$$\mathcal{T}^2 = \operatorname{im} M_i + \operatorname{im} X_i + \operatorname{im}(X^2)_i.$$

Since X is a distance-information preserving matrix with respect to a connected graph and $\operatorname{dist}(i, V_O) > 1$, there is vertex j such that $(X^2)_{i,j} \neq 0$ while $M_{i,j} = 0$ and $X_{i,j} = 0$. Hence, $\mathcal{T}^2$ is strictly larger than $\mathcal{T}^1$, and we can continue with the algorithm. By applying the same arguments repeatedly, we obtain (7.4a). In addition, it follows from (7.5) and (7.6) that

$$NM_i = NX_i = \cdots = N(X^{d_i-1})_i = 0.$$

Furthermore, we have

$$N(X^{d_i})_i \neq 0.$$

As such, (7.4b) holds. ■

The subspaces $\mathcal{T}_i^*$ are (N, X)-invariant and contain $\operatorname{im}(M_i)$ by definition, and so they satisfy conditions C1 and C2 of the geometric version of the fault detection and identification problem discussed in section 7.3. The result of Lemma 7.4 enables us to device a purely graph-theoretical sufficient condition for the output separability condition C3. To formulate the graph-theoretical condition, we need to introduce some nomenclature.

Given a simple graph $G = (V, E)$ and a pair (V_F, V_O), define

$$W_O = \{j \in V_O \mid \operatorname{dist}(i, j) = \operatorname{dist}(i, V_O) \text{ for some } i \in V_F\}.$$

In other words, W_O consists of the observer vertices that are the closest to one of the faultable vertices. Now, we define the bipartite graph $G_{OF} = (W_O, V_F, E_{OF})$ by

$$\{j, i\} \in E_{OF} \Leftrightarrow j \in W_O,\ i \in V_F,\ \operatorname{dist}(i, j) = \operatorname{dist}(i, V_O).$$

With these preparations, we are ready to state the graph-theoretical sufficient condition for the output separability requirement C3 for the class of distance-information preserving matrices.

Theorem 7.5 *Consider the system (7.1) for a simple and connected graph $G = (V, E)$ with faultable vertices V_F and observer vertices V_O. Then the family of subspaces $\{\mathcal{T}_i^*\}_{i=1}^q$ is output separable for any distance-information preserving matrix $X \in \mathcal{Q}(G)$ if the bipartite graph G_{OF} admits a constrained q-matching.*

Proof. For a given distance-information preserving matrix $X \in \mathcal{Q}(G)$, let

$$R = \begin{bmatrix} N(X^{d_1})_1 & N(X^{d_2})_2 & \cdots & N(X^{d_q})_q \end{bmatrix},$$

where $d_i = \text{dist}(i, V_O)$. Lemma 7.4 gives us that

$$N\mathcal{T}_i^* = \operatorname{im} N(X^{d_i})_i,$$

which implies that the output separability of the subspaces $\{\mathcal{T}_i^*\}_{i=1}^q$ is equivalent to the condition that

$$\operatorname{rank} R = q. \tag{7.7}$$

Observe that the jth entry of the column vector $N(X^{d_i})_i$ is nonzero if and only if $\text{dist}(i, j) = d_i = \text{dist}(i, V_O)$. Furthermore, note that the vertices in W_O correspond to the nonzero rows of the matrix R. This means that the matrix obtained from R by discarding the zero rows belongs to the pattern class $\mathcal{P}(G_{OF})$. Since the bipartite graph G_{OF} admits a constrained q-matching, it follows from Theorem 7.1 that $\operatorname{rank} R = q$. Consequently, the family $\{\mathcal{T}_i^*\}_{i=1}^q$ is output separable. ■

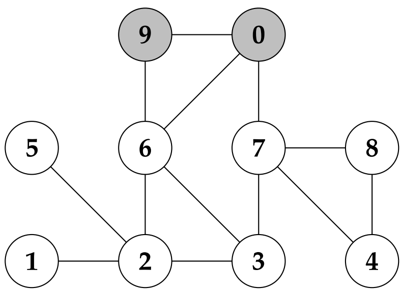

Figure 7.3: A 10-vertex graph.

Next we illustrate the result of Theorem 7.5 by means of an example.

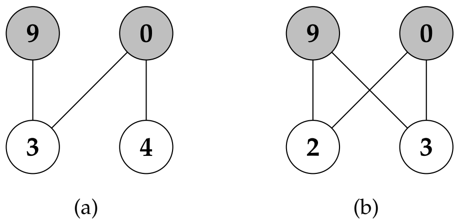

Figure 7.4: The graph G_{OF} for two different choices of observer vertices for the graph in Figure 7.3; (a) $V_F = \{0, 9\}$ and $V_O = \{3, 4\}$ and (b) $V_F = \{0, 9\}$ and $V_O = \{2, 3\}$.

Example 7.6 Consider the graph depicted in Figure 7.3. First, we choose $V_F = \{0, 9\}$ and $V_O = \{3, 4\}$. Note that $\text{dist}(0, V_O) = \text{dist}(9, V_O) = 2$ and $W_O = \{3, 4\}$. For this case, the bipartite graph G_{OF} is depicted in Figure 7.4a in which the observer vertices are indicated by octagons. The 2-matching $\{\{0, 4\}, \{9, 3\}\}$ is a constrained matching. As such, Theorem 7.5 concludes that the output separability condition C3 holds for this case, for any distance-information preserving matrix $X \in \mathcal{Q}(G)$.

Now, take the same graph and faultable vertices but change the set of observer vertices to $V_O = \{2, 3\}$. Again, we have $\text{dist}(0, V_O) = \text{dist}(9, V_O) = 2$, but this time $W_O = \{2, 3\}$. The new bipartite graph G_{OF} is depicted in Figure 7.4b. There are two 2-matchings, namely $\{\{0, 3\}, \{9, 2\}\}$ and $\{\{0, 4\}, \{9, 3\}\}$. As such, neither of them is a constrained matching and we cannot conclude whether the output separability condition is satisfied by employing Theorem 7.5.

The result of Theorem 7.5 can also be used for the design of systems on graphs: given a set of faultable vertices V_F, one way to guarantee output separability is to place sensors at certain (non-faultable) observer vertices so that the matching condition of Theorem 7.5 is satisfied.

The next example illustrates a pathological case for which the matching condition of Theorem 7.5 is not satisfied for any choice of observer vertices.

Example 7.7 Consider the graph depicted in Figure 7.3. Suppose that $V_F = \{1, 5\}$. Since $\text{dist}(1, i) = \text{dist}(5, i)$ for any vertex $i \notin V_F$, there is no choice of V_O for which the matching condition of Theorem 7.5 can be satisfied.

Based on this example, we can prove the following lemma, which can be seen as a necessary condition for output separability for the class of distance-information preserving matrices.

Lemma 7.8 *Consider the system (7.1) for a simple and connected graph $G = (V, E)$ with faultable vertices V_F and observer vertices V_O. Let $\{i, j\} \subseteq V_F$. Suppose that* $\mathrm{dist}(i, k) = \mathrm{dist}(j, k)$ *for any $k \notin \{i, j\}$. Then there exists at least one choice of a distance-information preserving matrix X with respect to the graph $G = (V, E)$ such that the subspaces $N\mathcal{T}_i^*$ and $N\mathcal{T}_j^*$ coincide.*

Proof. Let A denote the adjacency matrix corresponding to the graph $G = (V, E)$, then A is a distance-information preserving matrix with respect to the graph G. As $\mathrm{dist}(i, k) = \mathrm{dist}(j, k)$ for any $k \notin \{i, j\}$, we have that $A_{i,k} = A_{j,k}$ for any $k \notin \{i, j\}$. Moreover, we have

$$(A^\ell)_{i,k} = (A^\ell)_{j,k}$$

for any integer $\ell \geqslant 1$ and $k \notin \{i, j\}$, which implies that

$$N(A^\ell)_i = N(A^\ell)_j$$

for all $\ell \geqslant 1$. Then, Lemma 7.4 implies that

$$N\mathcal{T}_i^* = N\mathcal{T}_j^*.$$

As such, the output separability requirement is not satisfied by the choice of $X = A$. ■

7.5 CONCLUSIONS

In this chapter we have studied the fault detection and isolation problem for systems defined over graphs. First, we have provided a characterization of the so-called conditioned invariant subspaces of such systems with the distance-information preservation property. These subspaces play a major role in the analysis of fault detection as well as design of fault detectors. Based on this characterization, we have presented graph-theoretical sufficient conditions for the so-called output separability requirement that is the crux of the fault detection problem in the setting of geometric control. The graph-theoretical sufficient condition was illustrated on two examples. Based on another example, we have also presented a condition under which the output separability fails for the class of distance-information preserving matrices.

Investigating sharper sufficient conditions, devising an observer vertex selection method and formulating conditions that would guarantee the asymptotic stability of fault detectors are among the future research problems.

8 CONSENSUS DYNAMICS WITH ARBITRARY SIGN-PRESERVING NONLINEARITIES

ABSTRACT: *This chapter studies consensus problems for multi-agent systems defined on directed graphs where the consensus dynamics involves nonlinear and discontinuous functions. Sufficient conditions, involving the nonlinear functions and the topology of the underlying graph, for the agents to converge to consensus are provided. This chapter is based on the paper [Wei et al., 2016].*

8.1 INTRODUCTION

Apart from the popular linear consensus protocols, nonlinear agreement protocols have recently attracted the attention of many researchers. The nonlinear consensus protocols may arise due to the nature of the controller, see e.g. [Jafarian and De Persis, 2015; Saber and Murray, 2003], or may describe the physical coupling existing in the network, see e.g. [Bürger et al., 2014; Monshizadeh and De Persis, 2015]. In this chapter, we consider a general nonlinear consensus protocol. The topology among the agents is assumed to be a directed graph containing a directed spanning tree, which for the linear consensus protocol is known to be a sufficient and necessary condition for reaching state consensus.

The related works to this chapter can be divided into two categories, depending on whether the dynamical systems are continuous or not. For the case of continuous dynamical systems, closely related to this chapter are [Papachristodoulou et al., 2010] and [Lin et al., 2007]. In [Papachristodoulou et al., 2010], a general first-order consensus protocol with a continuous nonlinear function is considered for the case that there is a delay in the communication. In [Lin et al., 2007], the authors considered a nonlinear consensus protocol with Lipschitz continuous functions, under a switching topology. For the case of discontinuous dynamical system, [Cortés, 2006] is one of the major motivations of this chapter. Nonlinearities of the form of sign functions were considered in [Cortés, 2006], where the notion of Filippov solutions is employed. However, in order to guarantee the conclusion about the second network consensus protocol in [Cortés, 2006], more precise conditions turn out to be necessary.

This is formulated as the main result in Section 8.3.2. In [De Persis and Frasca, 2013], the authors considered a similar control protocol as in [Cortés, 2006] in a hybrid dynamical systems framework with a self-triggered communication policy, which avoids the notion of Filippov solutions. In addition, in [De Persis and Frasca, 2013] practical consensus is considered, that is, consensus within a predefined margin. The results presented in [Cortés, 2006; De Persis and Frasca, 2013] are restricted to undirected graphs. In [Dimarogonas and Johansson, 2010], the authors considered quantized communication protocols within the framework of hybrid dynamical systems, without using the notion of Filippov solutions. In this chapter we will consider a general nonlinear consensus protocol which incorporates the corresponding models in the previous works, and analyze the asymptotic stability of Filippov solutions.

The contribution of this chapter is to provide a uniform framework to analyze the asymptotic convergence towards consensus of a first-order consensus protocol for a very general class of discontinuous nonlinear functions, under the weakest fixed topology assumption, i.e., a directed graph containing a directed spanning tree. The analysis is conducted with the notion of Filippov solutions, and generalizes and corrects the second network consensus protocol in Cortés [2006]

The structure of the chapter is as follows. In Section 8.2, we introduce some terminology and notation in the context of graph theory and stability analysis of discontinuous dynamical systems. The main results are presented in Theorem 8.7 and Theorem 8.18 in Section 8.3. The general problem is introduced in Section 8.3.1, whereafter in Sections 8.3.2 and 8.3.3, two important subcases are considered. These results are then combined in Section 8.3.4.

8.2 PRELIMINARIES AND NOTATIONS

In this section we briefly review some notions from graph theory, and give some definitions and notation regarding Filippov solutions.

Let $G = (V, E, A)$ be a weighted digraph with node set $V = \{v_1, \dots, v_n\}$, edge set $E \subseteq V \times V$, and weighted adjacency matrix $A = [a_{ij}]$ with nonnegative adjacency elements a_{ij}. An edge of G is denoted by $e_{ij} := (v_i, v_j)$ and we write $\mathcal{I} = \{1, 2, \dots, n\}$. The adjacency elements a_{ij} are associated with the edges of the graph in the following way: $a_{ij} > 0$ if and only if $e_{ji} \in E$. Moreover, $a_{ii} = 0$ for all $i \in \mathcal{I}$. For undirected graphs, $A = A^\top$.

The set of neighbors of node v_i is denoted by $N_i := \{v_j \in V : e_{ji} \in E\}$. For each node v_i, its in-degree and out-degree are defined as

$$\deg_{\text{in}}(v_i) = \sum_{j=1}^{n} a_{ij}, \qquad \deg_{\text{out}}(v_i) = \sum_{j=1}^{n} a_{ji}.$$

The degree matrix of the digraph G is a diagonal matrix Δ where $\Delta_{ii} = \deg_{in}(v_i)$. The *graph Laplacian* is defined as

$$L = \Delta - A$$

and satisfies $L\mathbb{1} = 0$, where $\mathbb{1}$ is the n-vector containing only ones. We say that a node v_i is *balanced* if its in-degree and out-degree are equal. The graph G is called balanced if all of its nodes are balanced or, equivalently, if $\mathbb{1}^\top L = 0$.

A directed path from node v_i to node v_j is a chain of edges from E such that the first edge starts from v_i, the last edge ends at v_j and every edge in between starts where the previous edge ends. If for every two nodes v_i and v_j there is a directed path from v_i to v_j, then the graph G is called *strongly connected*. A subgraph $G' = (V', E', A')$ of G is called a *directed spanning tree* for G if $V' = V$, $E' \subseteq E$, and for every node $v_i \in V'$ there is exactly one node v_j such that $e_{ji} \in E'$, except for one node, which is called the root of the spanning tree. Furthermore, we call a node $v \in V$ a *root* of G if there is a directed spanning tree for G with v as a root. In other words, if v is a root of G, then there is a directed path from v to every other node in the graph. A digraph G is called *weakly connected* if G^o is connected, where G^o is the undirected graph obtained from G by ignoring the orientation of the edges.

The saturation function $\text{sat}(x\,; u^-, u^+) : \mathbb{R}^n \to \mathbb{R}^n$ is defined component-wisely as

$$\text{sat}(x\,; u^-, u^+)_i = \begin{cases} u_i^- & \text{if } x_i < u_i^-, \\ x_i & \text{if } x_i \in [u_i^-, u_i^+], \quad i \in \mathcal{I}, \\ u_i^+ & \text{if } x_i > u_i^+, \end{cases} \tag{8.1}$$

where u^- and u^+ are n-vectors containing the lower and upper bounds respectively. The sign function $\text{sign}(x) : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\text{sign}(x)_i = \begin{cases} -1 & \text{if } x_i < 0, \\ 0 & \text{if } x_i = 0, \quad i \in \mathcal{I}. \\ 1 & \text{if } x_i > 0, \end{cases} \tag{8.2}$$

With $\mathbb{R}_{-}$, $\mathbb{R}_{+}$ and $\mathbb{R}_{\geqslant 0}$ we denote the sets of negative, positive and nonnegative real numbers respectively. The vectors $e_1, e_2, \ldots, e_n$ denote the canonical basis of $\mathbb{R}^n$. The ith row and jth column of a matrix M are denoted by $M_{i\bullet}$ and $M_{\bullet j}$ respectively. For the empty set, we adopt the convention that $\max \varnothing = -\infty$.

In the rest of this section we give some definitions and notation regarding Filippov solutions (see, e.g., [Cortés, 2008]). Let F be a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and let $2^{\mathbb{R}^n}$ denote the collection of all subsets of $\mathbb{R}^n$. The map F is *essentially bounded* if there is a bound B such that $\|F(x)\|_2 < B$ for almost every $x \in \mathbb{R}^n$. The map F is *locally essentially bounded* if the restriction of F to every compact subset of $\mathbb{R}^n$ is essentially bounded. We define the *Filippov set-valued map* of F, denoted $\mathcal{F}[F] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, as

$$\mathcal{F}[F](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{\mathrm{co}}\{F(B(x,\delta)\backslash S)\}, \tag{8.3}$$

where $B(x,\delta)$ is the open ball centered at x with radius $\delta > 0$, S is a subset of $\mathbb{R}^n$, μ denotes the Lebesgue measure and $\overline{\mathrm{co}}$ denotes the convex closure. The zero measure set S is arbitrarily chosen. Hence, the set $\mathcal{F}[F](x)$ is independent of the value of $F(x)$. If F is continuous at x, then $\mathcal{F}[F](x)$ contains only the point $F(x)$. A *Filippov solution* of the differential equation $\dot{x}(t) = F(x(t))$ on $[0, t_1] \subset \mathbb{R}$ is an absolutely continuous function $x : [0, t_1] \to \mathbb{R}^n$ that satisfies the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[F](x(t)) \tag{8.4}$$

for almost all $t \in [0, t_1]$.

Let f be a map from $\mathbb{R}^n$ to $\mathbb{R}$. The right directional derivative of f at x in the direction of $v \in \mathbb{R}^n$ is defined as

$$f'(x; v) = \lim_{h \to 0^+} \frac{f(x+hv) - f(x)}{h},$$

when this limit exists. The generalized derivative of f at x in the direction of $v \in \mathbb{R}^n$ is given by

$$\begin{aligned} f^o(x; v) &= \limsup_{\substack{y \to x \\ h \to 0^+}} \frac{f(y+hv) - f(y)}{h} \\ &= \lim_{\substack{\delta \to 0^+ \\ \epsilon \to 0^+}} \sup_{\substack{y \in B(x,\delta) \\ h \in (0,\epsilon)}} \frac{f(y+hv) - f(y)}{h}. \end{aligned}$$

We call the function f *regular* at x if $f'(x;v)$ and $f^o(x;v)$ are equal for all $v \in \mathbb{R}^n$. For example, convex functions are regular, see e.g.,[Clarke, 1990].

If $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, then its *generalized gradient* $\partial f : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a set-valued function defined by

$$\partial f(x) := \text{co}\{\lim_{i\to\infty} \nabla f(x_i) : x_i \to x, x_i \notin S \cup \Omega_f\}, \tag{8.5}$$

where ∇ denotes the gradient operator, $\Omega_f \subset \mathbb{R}^n$ denotes the set of points where f fails to be differentiable, and $S \subset \mathbb{R}^n$ is a set of measure zero that can be chosen arbitrarily to simplify the computation. The resulting set $\partial f(x)$ is independent of the choice of S [Clarke, 1990].

Given a set-valued map $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, the *set-valued Lie derivative* $\tilde{\mathcal{L}}_{\mathcal{F}} f : \mathbb{R}^n \to 2^{\mathbb{R}}$ of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to $\mathcal{F}$ at x is defined as

$$\begin{aligned}\tilde{\mathcal{L}}_{\mathcal{F}} f(x) := \{a \in \mathbb{R} \,|\exists v \in \mathcal{F}(x) \text{ such that}\\ \zeta^{\mathsf{T}} v = a \text{ for all } \zeta \in \partial f(x)\}.\end{aligned} \tag{8.6}$$

A Filippov solution $t \mapsto x(t)$ is *maximal* if it cannot be extended forward in time, that is, if $t \to x(t)$ is not the result of the truncation of another solution with a larger interval of definition. Since the Filippov solutions of a discontinuous system (8.4) are not necessarily unique, we need to specify two types of invariant set. A set $\mathcal{R} \subset \mathbb{R}^n$ is called *weakly invariant* for (8.4) if, for each $x_0 \in \mathcal{R}$, at least one maximal solution of (8.4) with initial condition x_0 is contained in $\mathcal{R}$. Similarly, $\mathcal{R} \subset \mathbb{R}^n$ is called *strongly invariant* for (8.4) if, for each $x_0 \in \mathcal{R}$, every maximal solution of (8.4) with initial condition x_0 is contained in $\mathcal{R}$. For more details, see [Cortés, 2008].

8.3 MAIN RESULTS

8.3.1 *Problem formulation*

In this chapter we consider a network of n agents, who communicate according to a communication topology given by a weighted directed graph $G = (V, E, A)$. In this network, agent i receives information from agent j if and only if there is an edge from node v_j to node v_i in the graph G. We denote the state of agent i at time t as $x_i(t) \in \mathbb{R}$, and consider the following dynamics for agent i

$$\dot{x}_i = f_i(\sum_{j=1}^{n} a_{ij} g_{ij}(x_j - x_i)) =: h_i(x), \tag{8.7}$$

where f_i and g_{ij} are functions from $\mathbb{R}$ to $\mathbb{R}$ and a_{ij} are the elements of the adjacency matrix A.

Each function f_i describes how agent i handles incoming information, while the functions g_{ij} are concerned with the flow of information along the edges of the graph G. All these functions are nonlinear and may have discontinuities, but we will use the concept of sign-preserving functions.

Definition 8.1 We say that a function $\varphi : \mathbb{R} \to \mathbb{R}$ is *sign preserving* if $\varphi(0) = 0$ and for each $y \in \mathbb{R} \setminus \{0\}$ we have both $y\varphi(y) > 0$ and $\min y\mathcal{F}[\varphi](y) > 0$.

Examples of sign-preserving functions are the signum function sign and the saturation function sat. If a function φ has only finitely many discontinuities, e.g. when it is piecewise continuous, then the condition $y\mathcal{F}[\varphi](y) > 0$ only needs to be checked for its discontinuity points. The second condition in Definition 8.1, $\min y\mathcal{F}[\varphi](y) > 0$ for all $y \in \mathbb{R} \setminus \{0\}$, will be illustrated in Example 8.4.

Throughout this chapter, we assume the following.

Assumption 8.2 The functions f_i and g_{ij} are sign preserving, Lebesgue measurable, and locally essentially bounded.

To handle possible discontinuities in the right-hand side of (8.7), we consider Filippov solutions of the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[h](x(t)). \tag{8.8}$$

The existence of a Filippov solution for each initial condition is guaranteed by the Lebesgue measurability and the local essential boundedness of the functions f_i and g_{ij} in Assumption 8.2. In this chapter we assume the completeness of Filippov solutions of (8.8) for any initial condition. Notice that when the functions h_i are globally bounded, e.g., if f_i and g_{ij} are chosen as signum or saturation functions, then the completeness of Filippov solution of (8.8) is guaranteed by Theorem 1 in §7 of Chapter 2 in [Filippov, 1988]. Moreover, by property 3 of Theorem 1 in [Paden and Sastry, 1987], we have

$$\mathcal{F}[h](x(t)) \subset \bigtimes_{i=1}^{n} \mathcal{F}[h_i](x(t)) \tag{8.9}$$

where $\bigtimes$ denotes the Cartesian product.

The agents of the network are said to achieve *consensus* if they all converge to the same value, that is,

$$\lim_{t\to\infty} x(t) = \eta \mathbb{1}$$

for some *constant* $\eta \in \mathbb{R}$, where $x(t) = [x_1(t), \ldots, x_n(t)]^\top$ is a solution of (8.7) with $x(0) = x_0$. It is well known that if all functions f_i and g_{ij} are the identity function, in which case (8.7) boils down to the linear consensus protocol, then the agents will achieve consensus if and only if the graph G contains a directed spanning tree [Agaev and Chebotarev, 2005; Ren et al., 2005]. In this chapter, we investigate the consensus problem for general functions f_i and g_{ij} satisfying Assumption 8.2. First, in Section 8.3.2, we consider the special case that the functions g_{ij} are equal to the identity function, that is $\dot{x}_i = f_i(\sum_{j=1}^n a_{ij}(x_j - x_i))$. Thereafter, in Section 8.3.3, we consider the case where the functions f_i are the identity function, that is $\dot{x}_i = \sum_{j=1}^n a_{ij} g_{ij}(x_j - x_i)$. Finally, in section 8.3.4, we will combine these results.

The following examples motivate the sign-preserving condition by showing what happens if the functions f_i and g_{ij} do not satisfy this property.

Example 8.3 Consider the following system defined on the graph given in Fig. 8.5a

$$\dot{x}_1 = f_1(0) \tag{8.10a}$$
$$\dot{x}_2 = f_2(x_1 - x_2), \tag{8.10b}$$

with $f_i(y) = \mathrm{sat}(y; 0, 1)$ for $i = 1, 2$. Notice that f_i satisfies $y f_i(y) = 0$ for all $y < 0$, and hence f_i is not sign preserving. In this case the existence of a directed spanning tree is not a sufficient condition for convergence to consensus. Indeed, if the initial condition satisfies $x_2(0) > x_1(0)$, then $x_1(t) = x_1(0)$ and $x_2(t) = x_2(0)$ for all $t \geqslant 0$. Hence, the agents do not reach consensus.

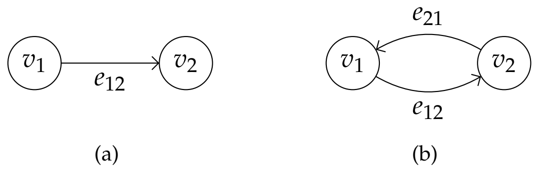

Figure 8.5: Two digraphs with two nodes for Examples 8.3, 8.10 and 8.11. (a) Digraph with spanning tree and (b) Strongly connected digraph.

Example 8.4 Consider the system (8.10) defined on the digraph in Fig. 8.5a with f_i given by

$$f_i(y) = \begin{cases} y+1 & \text{if } y < -1 \\ y & \text{if } y \in [-1, 1], \quad i = 1, 2. \\ y-1 & \text{if } y > 1 \end{cases} \tag{8.11}$$

Then the function f_i satisfies $f_i(0) = 0$ and $yf_i(y) > 0$ for all $y \neq 0$. However, since $\mathcal{F}[f_i](1) = [0, 1]$ and $\mathcal{F}[f_i](-1) = [-1, 0]$, we have that $\min y\mathcal{F}[f_i](y) > 0$ is not satisfied for $y = \pm 1$. Hence, f_i is not sign preserving. Consider the point $x^* = [0, 1]^{\mathsf{T}}$, we have

$$\mathcal{F}[h](x^*) = \overline{\text{co}}\{[0, -1]^{\mathsf{T}}, [0, 0]^{\mathsf{T}}\},$$

which contains the point $[0, 0]^{\mathsf{T}}$. Consequently, x^* is an equilibrium point of the differential inclusion $\dot{x}(t) \in \mathcal{F}[h](x(t))$. For example, the trajectory

$$x_1(t) = 0, \ x_2(t) = 1 + e^{-t}$$

is a solution of (8.10) which converges to x^*. Therefore, the agents do not reach consensus.

8.3.2 *Node nonlinearity*

We first consider the system (8.7) where the functions g_{ij} are all the identity function, and focus our attention on the functions f_i, which describe how agent i handles the incoming information flow. In this case, the total dynamics of the agents can be written as

$$\dot{x} = f(-Lx), \tag{8.12}$$

where L is the graph Laplacian induced by the information flow digraph $G = (V, E, A)$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$f(y) = [f_1(y_1), f_2(y_2), \ldots, f_n(y_n)]^\mathsf{T}.$$

In this case we consider Filippov solutions of the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[h](x(t)), \tag{8.13}$$

where $h(x) = f(-Lx)$. Note that, since L is a singular matrix, we have $\mathcal{F}[h](x(t)) \neq \mathcal{F}[f](-Lx(t))$ in general.

The aim of this section is to investigate under which conditions the Filippov solutions of the system (8.13) achieve consensus. Because of possible discontinuity of the right-hand side of (8.12), there can be Filippov solutions of (8.13) that are unbounded. The following example illustrates this unwanted behavior.

Example 8.5 Consider a dynamical system (8.12) defined on an undirected graph with three nodes, as given in Fig. 8.6a, where the functions f_i are all given by the signum function:

$$\begin{aligned}
\dot{x}_1 &= \operatorname{sign}(x_2 + x_3 - 2x_1) \\
\dot{x}_2 &= \operatorname{sign}(x_1 + x_3 - 2x_2) \\
\dot{x}_3 &= \operatorname{sign}(x_1 + x_2 - 2x_3).
\end{aligned}$$

Suppose that at time t_0 we have $x(t_0) \in \operatorname{span}\{\mathbb{1}\}$, then

$$\mathcal{F}[h](x(t_0)) = \overline{\operatorname{co}}\,\{v_1, v_2, v_3, -v_1, -v_2, -v_3\}\,, \tag{8.14}$$

where $v_1 = [1, 1, -1]^\mathsf{T}$, $v_2 = [1, -1, 1]^\mathsf{T}$, and $v_3 = [-1, 1, 1]^\mathsf{T}$. Since $\sum_{i=1}^{3} \frac{1}{3} v_i = \frac{1}{3}\mathbb{1}$, we have that $\{\eta\mathbb{1} \mid \eta \in [-\frac{1}{3}, \frac{1}{3}]\} \subset \mathcal{F}[h](x(t_0))$. Hence, any function $x(t) = \eta(t)\mathbb{1}$ with $\eta(t)$ differentiable almost everywhere and satisfying $\dot{\eta}(t) \in [-\frac{1}{3}, \frac{1}{3}]$ is a Filippov solution for this system. For example, $x(t) = \frac{1}{3}\mathbb{1}$ and $x(t) = \frac{1}{3}\sin(t)\mathbb{1}$ are Filippov solutions for this system that exhibit sliding consensus.

The undesirable behavior $x(t) = \eta(t)\mathbb{1}$ with $\eta(t)$ a nonconstant function in the previous example will be called *sliding consensus*. Sliding consensus arises whenever $\eta\mathbb{1}$ is contained in $\mathcal{F}[h](\alpha\mathbb{1})$ for some scalars $\eta \neq 0$ and sufficiently many α. Note that this example shows that for the validity of Theorem 11 in [Cortés, 2006] we need extra conditions; a counter example to Theorem 11 *(i)* in Cortés [2006] can be constructed similarly to

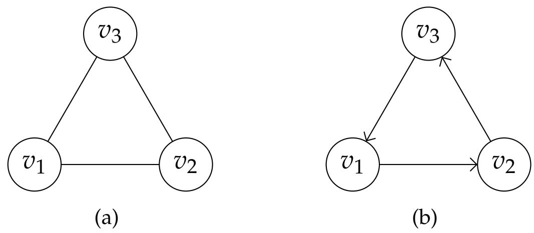

Figure 8.6: Two graphs with three nodes, used in Examples 8.5, 8.9, 8.13 and 8.16. (a) Undirected graph and (b) Directed graph.

Example 8.5. In fact, it will turn out that the sliding consensus can be excluded by replacing the signum function for at least one node by a function that is continuous at the origin. This motivates the introduction of the following subsets of the index set $\mathcal{I}$ corresponding to the digraph G:

$$\begin{aligned} \mathcal{I}^r &= \{i \in \mathcal{I} \mid v_i \text{ is a root of } G\}, \\ \mathcal{I}^c &= \{i \in \mathcal{I} \mid f_i \text{ is continuous at the origin}\}. \end{aligned}$$

Before we present the main result of this section, we first state a preparatory lemma [Clarke, 1990, Prop. 2.2.6 and Prop. 2.3.6].

Lemma 8.6 *The following functions are regular and Lipschitz continuous,*

$$V(x) := \max_{i \in \mathcal{I}} x_i, \qquad W(x) := -\min_{i \in \mathcal{I}} x_i. \tag{8.15}$$

Theorem 8.7 *Consider system (8.13) defined on a digraph $G = (V, E, A)$. If one of the following three conditions holds, i.e.,*

(i) $\mathcal{I}^c \cap \mathcal{I}^r$ *is not empty,*

(ii) $|\mathcal{I}^r| = 1$,

(iii) $|\mathcal{I}^r| = 2$, *$f_i(0^-)$ and $f_i(0^+)$ exist, and $f_i(0^-) = -f_i(0^+)$ for $i \in \mathcal{I}^r$,*

then all the trajectories of system (8.13) achieve consensus asymptotically, for any initial condition. Furthermore, they will remain in the set $[\min_i x_i(0), \max_i x_i(0)]^n$ for all $t \geqslant 0$.

Proof. Notice that in all three cases the index set $\mathcal{I}^r$ is nonempty, which implies that the graph G contains a directed spanning tree. Condition *(i)* implies that the digraph G has a root v_i for which f_i is continuous at the origin.

Consider candidate Lyapunov functions V and W as given in (8.15). By Lemma 8.6, the functions V and W are regular and Lipschitz continuous. Let $x(t)$ be a trajectory of (8.13) and define

$$\alpha(t) = \{k \in \mathcal{I} \mid x_k(t) = V(x(t))\}.$$

The generalized gradient of V is given as [Clarke, 1990, Example 2.2.8]

$$\partial V(x(t)) = \mathrm{co}\{e_k \in \mathbb{R}^n \mid k \in \alpha(t)\}. \tag{8.16}$$

Let Ψ be defined as

$$\Psi = \{t \geqslant 0 \mid \text{both } \dot{x}(t) \text{ and } \frac{d}{dt}V(x(t)) \text{ exist}\}. \tag{8.17}$$

Since x is absolutely continuous and V is locally Lipschitz, we have that $\Psi = \mathbb{R}_{\geqslant 0} \setminus \bar{\Psi}$ for a set $\bar{\Psi}$ of measure zero. By Lemma 1 in [Bacciotti and Ceragioli, 1999], we have

$$\frac{d}{dt}V(x(t)) \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) \tag{8.18}$$

for all $t \in \Psi$ and hence that the set $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ is nonempty for all $t \in \Psi$. For $t \in \bar{\Psi}$, we have that $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ is empty, and hence $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) < 0$. For $t \in \Psi$, let $a \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$. By definition, there exists a $\nu^a \in \mathcal{F}[h](x(t))$ such that $a = \nu^a \cdot \zeta$ for all $\zeta \in \partial V(x(t))$. Consequently, by choosing $\zeta = e_k$ for $k \in \alpha(t)$, we observe that ν^a satisfies

$$\nu_k^a = a \qquad \forall k \in \alpha(t). \tag{8.19}$$

Next, we want to show that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) \leqslant 0$ for all $t \in \Psi$ by considering two possible cases: $\mathcal{I}^r \nsubseteq \alpha(t)$ or $\mathcal{I}^r \subseteq \alpha(t)$.

If $\mathcal{I}^r \nsubseteq \alpha(t)$, then there exists an $i \in \mathcal{I}^r$ such that $x_i(t) < V(x(t))$. Furthermore, since v_i is a root, we can choose an index $j \in \alpha(t)$ such that the shortest path from v_i to v_j has the least number of edges. By our choice of j, there is at least one edge $e_{kj} \in E$ such that $x_k(t) < x_j(t)$, which implies that we have $-L_{j\bullet}x(t) < 0$. Moreover, the existence of an edge e_{kj} implies that $\operatorname{rank} L_{j\bullet} = 1$, which together with property 4 of Theorem 1 in [Paden and Sastry, 1987] gives us

$$\mathcal{F}[h_i](x(t)) = \mathcal{F}[f_j](-L_{j\bullet}x(t)). \tag{8.20}$$

By the sign-preserving property of f_j and $-L_{j\bullet}x(t) < 0$, we have that $\mathcal{F}[h_j](x(t)) \subset \mathbb{R}_-$. By (8.9), we find that $\nu_j < 0$ for any $\nu \in \mathcal{F}[h](x(t))$. Using observation (8.19) for $k = j$, we see that every $a \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ satisfies $a < 0$. By the fact that $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ is a closed set, we have $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) < 0$.

If $\mathcal{I}^r \subseteq \alpha(t)$, we will consider the conditions *(i)*, *(ii)* and *(iii)* separately and prove that $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) = \{0\}$. First, we note that if a node v_k is a root, then $e_{jk} \in E$ implies that v_j is a root as well, and hence we have

$$(-Lx)_k = \sum_{j\in\mathcal{I}} a_{kj}(x_j - x_k) = \sum_{j\in\mathcal{I}^r} a_{kj}(x_j - x_k). \tag{8.21}$$

(i) In this case $\mathcal{I}^c \cap \mathcal{I}^r \subseteq \alpha(t)$. For any $i \in \mathcal{I}^c \cap \mathcal{I}^r$, we have that f_i is continuous at 0 and satisfies $f_i(0) = 0$. This implies that any $\nu \in \mathcal{F}[h](x(t))$ satisfies $\nu_i = 0$. Using observation (8.19), we can conclude that $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) = \{0\}$.

(ii) Let $\mathcal{I}^r = \{i\}$. Since there is only one root in this case, namely v_i, we have $L_{i\bullet} = 0$ and hence $f_i((Lx(t))_i) = f_i(0) = 0$ for all t. Consequently, each $\nu \in \mathcal{F}[h](x(t))$ satisfies $\nu_i = 0$. Using observation (8.19) again, we see that $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) = \{0\}$.

(iii) Let $\mathcal{I}^r = \{i, j\}$. By (8.21), the dynamics of x_i and x_j are given as $\dot{x}_i = f_i(a_{ij}(x_j - x_i))$ and $\dot{x}_j = f_j(a_{ji}(x_i - x_j))$ respectively.

Since $\mathcal{I}^r \subseteq \alpha(t)$, we have $x_i(t) = x_j(t)$ and hence any $\nu \in \mathcal{F}[h](x(t))$ satisfies

$$\begin{aligned}\begin{bmatrix}\nu_i\\ \nu_j\end{bmatrix} &\subseteq \overline{\mathrm{co}}\{\begin{bmatrix}f_i(0^-)\\ f_j(0^+)\end{bmatrix}, \begin{bmatrix}f_i(0^+)\\ f_j(0^-)\end{bmatrix}\}\\ &= \overline{\mathrm{co}}\{\begin{bmatrix}f_i(0^-)\\ f_j(0^+)\end{bmatrix}, -\begin{bmatrix}f_i(0^-)\\ f_j(0^+)\end{bmatrix}\},\end{aligned} \tag{8.22}$$

where the last equality is implied by condition *(iii)*. Moreover, by condition *(iii)*, the convex set given in (8.22) is a line segment that only crosses $\mathrm{span}\{[1, 1]^\mathsf{T}\}$ in the origin. This implies that any $\nu \in \mathcal{F}[h](x(t))$ with $\nu_i = \nu_j$ must satisfy $\nu_i = \nu_j = 0$. Using $\partial V(x(t)) = \mathrm{co}\{e_i, e_j \in \mathbb{R}^n\}$ and (8.19), we see that $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) = \{0\}$.

Define $\beta(t) = \{i \in \mathcal{I} \mid x_i(t) = -W(x(t))\}$. By using similar computations and observations as above, we find that

$\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) < 0$ if $\mathcal{I}^r \nsubseteq \beta(t)$, and $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) \leqslant 0$ if $\mathcal{I}^r \subseteq \beta(t)$.

We conclude that $V(x(t))$ and $W(x(t))$ are not increasing along the trajectories $x(t)$ of the system (8.13). Hence, the trajectories are bounded and remain in $[\min_i x_i(0), \max_i x_i(0)]^n$ for all $t \geqslant 0$. Therefore, for any $N \in \mathbb{R}_+$, the set $S_N = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant N\}$ is strongly invariant for (8.13). By Theorem 2 in [Cortés, 2008], we have that all solutions of (8.13) starting at S_N converge to the largest weakly invariant set M contained in

$$S_N \cap \overline{\{x \mid 0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x)\}} \cap \overline{\{x \mid 0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x)\}}. \quad (8.23)$$

From the argument above we see that $0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$ is only possible if $\mathcal{I}^r \subseteq \alpha(t)$, and $0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t))$ can only happen if $\mathcal{I}^r \subseteq \beta(t)$. This implies that for every root v_i, the state x_i converges simultaneously to the maximum and to the minimum, i.e., the trajectories $x(t)$ of the system achieve consensus for any initial condition. ■

Remark 8.8 The set $\mathcal{I}^c$ collects all the functions which are continuous at the origin. In fact, this set can be enlarged such that it contains all the functions which are *essentially continuous* at the origin, i.e., $\operatorname{ess\,lim}_{x_i \to 0^-} f_i(x_i) = \operatorname{ess\,lim}_{x_i \to 0^+} f_i(x_i) = 0$ (for definitions see e.g. [Arutyunov, 2000; Chung and Walsh, 2006]). This can be done since in the definitions of both essential limits and Filippov set-valued map, any zero measure set can be excluded. For condition *(iii)* in Theorem 8.7, the same extension is possible; considering essential limits in stead of limits.

The conditions *(i)*, *(ii)* and *(iii)* in Theorem 8.7 all exclude the possibility of sliding consensus, and guarantee asymptotic consensus. The role of each condition will be illustrated in the following examples.

Example 8.9 Consider system (8.12) defined on the undirected graph in Fig. 8.6a, defined as

$$\begin{aligned} \dot{x}_1 &= f_1(x_2 + x_3 - 2x_1) \\ \dot{x}_2 &= f_2(x_1 + x_3 - 2x_2) \\ \dot{x}_3 &= f_3(x_1 + x_2 - 2x_3). \end{aligned}$$

Suppose that f_1 is continuous at the origin, so that condition *(i)* in Theorem 8.7 is satisfied. Then the sliding consensus is not a Filippov solution. Indeed, if at time t_0 we have $x(t_0) \in$

span$\{\mathbb{1}\}$, then the first component of the Filippov set-valued map $\mathcal{F}[h](x(t_0))$ is equal to $\{0\}$. This implies that $x(t) = x(t_0)$, for all $t \geqslant t_0$.

Example 8.10 Consider system (8.10) defined on the digraph in Fig. 8.5a. It satisfies condition *(ii)* of Theorem 8.7. Since $f_1(0) = 0$, the state of the root v_1 is constant. Consensus is achieved by the fact that f_2 is sign preserving.

Example 8.11 Consider system (8.12) defined on the digraph given in Fig. 8.5b with $a_{12} = a_{21} = 1$. The dynamics are given by

$$\begin{aligned}\dot{x}_1 &= f_1(x_2 - x_1)\\ \dot{x}_2 &= f_2(x_1 - x_2).\end{aligned}$$

First, we consider a case in which f_1 and f_2 satisfy condition *(iii)* of Theorem 8.7 and take $f_1 = f_2 = \text{sign}(\cdot)$. If the trajectory achieves consensus at time t, the Filippov set-valued map $\mathcal{F}[h](x(t))$ equals $\overline{\text{co}}\{[1,-1]^\top, [-1,1]^\top\}$, which intersects span$\{\mathbb{1}\}$ only at $[0,0]^\top$. Hence $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x) = \tilde{\mathcal{L}}_{\mathcal{F}[h]}W(x) = 0$, which implies that the trajectory remains constant, i.e., there is no sliding consensus.

Second, we consider a case in which $f_i(0^-) \neq -f_i(0^+)$ for $i = 1, 2$, which means that the conditions of Theorem 8.7 (and condition *(iii)* in particular) are not satisfied. In this case, sliding consensus can be a Filippov solution. For instance, take

$$f_i(x) = \begin{cases} 2 & \text{if } x > 0\\ 0 & \text{if } x = 0,\\ -1 & \text{if } x < 0\end{cases} \qquad i = 1, 2.$$

Suppose that at t_0 the state x achieves consensus. Then the Filippov set-valued map at $x(t_0)$ is $\overline{\text{co}}\{[-1,2]^\top, [2,-1]^\top\}$ which intersects span$\{\mathbb{1}\}$ at $[\frac{1}{2}, \frac{1}{2}]^\top$. Then $x(t) = \frac{1}{2}\mathbb{1}t + x(t_0)$ is a Filippov solution for $t \geqslant t_0$ that exhibits sliding consensus.

8.3.3 *Edge nonlinearity*

In this section we consider the case where the functions f_i are all the identity function, that is,

$$\begin{aligned}\dot{x}_i &= \sum_{j=1}^{N} a_{ij} g_{ij}(x_j - x_i) \\ &=: h_i(x), \quad i \in \mathcal{I}.\end{aligned} \tag{8.24}$$

We consider two cases, corresponding to the underlying graph $G = \{V, E\}$ being undirected or directed, starting with the undirected case. We introduce the following assumption on the functions g_{ij}.

Assumption 8.12 For all $e_{ji} \in E$, the right and left limits of g_{ij} and g_{ji} at the origin exist, and satisfy $g_{ij}(0^-) = -g_{ji}(0^+)$.

To illustrate the need of Assumption 8.12, we give the following example.

Example 8.13 If $g_{ij}(0^-) \neq -g_{ji}(0^+)$, then sliding consensus may occur. For instance, consider the system (8.24) defined on the undirected graph in Fig. 8.6a given by

$$\begin{aligned}
\dot{x}_1(t) &= g_{12}(x_2(t) - x_1(t)) + g_{13}(x_3(t) - x_1(t)) \\
\dot{x}_2(t) &= g_{21}(x_1(t) - x_2(t)) + g_{23}(x_3(t) - x_2(t)) \\
\dot{x}_3(t) &= g_{31}(x_1(t) - x_3(t)) + g_{32}(x_2(t) - x_3(t))
\end{aligned}$$

where

$$g_{ij}(x) = \begin{cases} 1.5 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -0.5 & \text{if } x < 0, \end{cases} \quad \forall e_{ji} \in E,$$

Suppose that at time t_0 the state satisfies $x(t_0) \in \operatorname{span}\{\mathbb{1}\}$, then $\mathcal{F}[h](x(t_0))$ is the closed convex hull of

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Hence, $\mathbb{1}$ is an element in $\mathcal{F}[h](x(t_0))$ and thus $x(t) = t\mathbb{1} + x(t_0)$ is a Filippov solution for $t > t_0$.

Next, we present the main result of this section.

Theorem 8.14 *Consider the dynamics (8.24) defined on a connected undirected graph. Suppose the functions g_{ij} satisfy Assumptions 8.2 and 8.12. Then the trajectories of the system (8.24) achieve consensus asymptotically.*

Proof. Consider the Lyapunov candidate functions V and W as defined in (8.15). We use the same notations as in the proof of Theorem 8.7. Similarly, as the proof of Theorem 8.7, we only prove that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) \leqslant 0$ for all $t \in \Psi$ where $\mathbb{R}_{\geqslant 0} \setminus \Psi$

is a set of measure zero and the set $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ is nonempty for all $t \in \Psi$.

By introducing the functions $\varphi_{ji}(x) = (e_j - e_i)^\mathsf{T} x$ for $i, j \in \mathcal{I}$, the function $h_i(x)$ in (8.24) can be rewritten as

$$h_i(x) = \sum_{j=1}^{n} a_{ij}(g_{ij} \circ \varphi_{ji})(x). \tag{8.25}$$

Then, using Theorem 1 in [Paden and Sastry, 1987], we see that the Filippov set-valued map $\mathcal{F}[h](x)$ satisfies

$$\begin{aligned} \mathcal{F}[h](x) &\subset \bigtimes_{i=1}^{n} \mathcal{F}[h_i](x) \\ &\subset \bigtimes_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\mathcal{F}[g_{ij} \circ \varphi_{ji}](x) \\ &= \bigtimes_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\mathcal{F}[g_{ij}](\varphi_{ji}(x)). \end{aligned} \tag{8.26}$$

By property 1 in Theorem 1 in [Paden and Sastry, 1987], for each g_{ij} there exists a set $N_{g_{ij}} \subset \mathbb{R}^n$ with $\mu(N_{g_{ij}}) = 0$ such that

$$\mathcal{F}[g_{ij}](z) = \mathrm{co}\{\lim_{k\to\infty} h(z^k) \mid \lim_{k\to\infty} z^k = z, z^k \notin N_{g_{ij}} \cup N'\} \tag{8.27}$$

for any set N' with $\mu(N') = 0$. Similarly, there exists a set $N_h \subset \mathbb{R}^n$ with $\cup_{e_{ij}\in E} N_{g_{ij}} \subset N_h$ and $\mu(N_h) = 0$ such that

$$\mathcal{F}[h](x(t)) = \mathrm{co}\{\lim_{k\to\infty} h(y^k) \mid \lim_{k\to\infty} y^k = x(t), y^k \notin N_h \cup S\}, \tag{8.28}$$

where $S = \{x \in \mathbb{R}^n \mid \exists i, j \in \mathcal{I}$ such that $x_i = x_j\}$, which has measure zero in $\mathbb{R}^n$. Notice that $\mathbb{R}^n \setminus S$ admits a partition $\mathbb{R}^n \setminus S = S_1 \cup S_2 \cup \cdots \cup S_{2^n}$, with $S_1, S_2, \ldots, S_{2^n}$ open sets satisfying $S_i \cap S_j = \varnothing$ for all $i \neq j$, such that within a fixed open set S_i, the components $y_1, y_2, \ldots, y_n$ of each vector $y \in S_i$ are all different and have the same fixed order.

Now, to study the right-hand side of (8.28), let t be a given time and let (y^k) be a sequence in $\mathbb{R}^n \setminus (N_h \cup S)$ that converges to $x(t)$ for which the limit $\tilde{h} := \lim_{k\to\infty} h(y^k)$ exists. Note that the existence of $\lim_{k\to\infty} h(y^k)$ means that all the components $\tilde{h}_i := h_i(y^k)$ have a limit. We will study the term $\sum_{i\in\alpha(t)} \tilde{h}_i$ in order to derive that $\sum_{i\in\alpha(t)} \nu_i \leqslant 0$ for each $\nu \in \mathcal{F}[h](x(t))$. For this, we first define two sets of edges, namely

$$\begin{aligned} E_1(t) &= \{e_{ij} \in E \mid i, j \in \alpha(t)\}, \\ E_2(t) &= \{e_{ij} \in E \mid i \in \alpha(t), j \notin \alpha(t)\}. \end{aligned} \tag{8.29}$$

The sequence (y^k) has a subsequence (y^{k_ℓ}) such that $y^{k_\ell} \in S_r$ for all ℓ for a fixed $r \in \{1, 2, \dots, 2^n\}$. For an edge $e_{ij} \in E_1$, we have $y_i^{k_\ell} - y_j^{k_\ell} \uparrow 0$ or $y_i^{k_\ell} - y_j^{k_\ell} \downarrow 0$, depending on the set S_r. Therefore, we have

$$\begin{aligned} \lim_{\ell\to\infty} \begin{bmatrix} g_{ij}(\varphi_{ji}(y^{k_\ell})) \\ g_{ji}(\varphi_{ij}(y^{k_\ell})) \end{bmatrix} &= \begin{bmatrix} g_{ij}(0^-) \\ g_{ji}(0^+) \end{bmatrix} \quad \text{or} \\ \lim_{\ell\to\infty} \begin{bmatrix} g_{ij}(\varphi_{ji}(y^{k_\ell})) \\ g_{ji}(\varphi_{ij}(y^{k_\ell})) \end{bmatrix} &= \begin{bmatrix} g_{ij}(0^+) \\ g_{ji}(0^-) \end{bmatrix}. \end{aligned} \tag{8.30}$$

Using Assumption 8.12, we see that in both cases we have

$$\lim_{\ell\to\infty} g_{ij}(\varphi_{ji}(y^{k_\ell})) + g_{ji}(\varphi_{ij}(y^{k_\ell})) = 0. \tag{8.31}$$

Now, we can write

$$\begin{aligned} \sum_{i\in\alpha(t)} \lim_{k\to\infty} h_i(y^k) &= \lim_{\ell\to\infty} \sum_{i\in\alpha(t)} \sum_{j=1}^{n} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) \\ &= \lim_{\ell\to\infty} \sum_{e_{ij}\in E_2} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) + \\ &\quad \lim_{\ell\to\infty} \sum_{e_{ij}\in E_1} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) \\ &= \lim_{\ell\to\infty} \sum_{e_{ij}\in E_2} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) + \\ &\quad \frac{1}{2} \sum_{e_{ij}\in E_1} \lim_{\ell\to\infty} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) + a_{ji} g_{ji}(\varphi_{ij}(y^{k_\ell})) \\ &= \lim_{\ell\to\infty} \sum_{e_{ij}\in E_2} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})), \end{aligned} \tag{8.32}$$

where the last two equalities are implied by the fact that the graph G is undirected and by equation (8.31). Next, we consider two possible cases: $x(t) \notin \operatorname{span}\{\mathbb{1}\}$, and $x(t) \in \operatorname{span}\{\mathbb{1}\}$.

First, we look at the case that $x(t) \notin \operatorname{span}\{\mathbb{1}\}$, in which case $E_2 \neq \varnothing$. For an edge $e_{ij} \in E_2$ we have $x_j < x_i$, and since g_{ij} is a sign-preserving function, this implies that $\mathcal{F}[g_{ij}](x_j - x_i) \subset \mathbb{R}_-$. As $y^{k_\ell} \notin N_{g_{ij}}$, all the accumulation points of the sequence $\{g_{ij}(\varphi_{ji}(y^{k_\ell}))\}$ belong to $\mathcal{F}[g_{ij}](x_j - x_i)$. Therefore, we see that

$$\sum_{i\in\alpha(t)} \lim_{k\to\infty} h_i(y^k) = \lim_{\ell\to\infty} \sum_{e_{ij}\in E_2} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) < 0,$$

i.e., $\sum_{i\in\alpha(t)} \tilde{h}_i < 0$. By equation (8.28), we can conclude that $\sum_{i\in\alpha(t)} \nu_i < 0$ for any $\nu \in \mathcal{F}[h](x(t))$. Hence, by observation (8.19), we have $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) \subset \mathbb{R}_{-}$. Since $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ is closed (see e.g. page 63 in [Cortés, 2008]), we have

$$\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) < 0.$$

Second, we consider the case that $x(t) \in \mathrm{span}\{\mathbb{1}\}$, in which case $E_1(t) = E$ and $E_2(t) = \varnothing$. In this case, equation (8.32) boils down to

$$\sum_{i\in\alpha(t)} \tilde{h}_i = \sum_{i\in\mathcal{I}} \lim_{k\to\infty} h_i(y^k) = 0. \tag{8.33}$$

By equation (8.28), we can conclude that $\sum_{i\in\mathcal{I}} \nu_i = 0$ for any $\nu \in \mathcal{F}[h](x(t))$. This implies that $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) = \{0\}$ since $\frac{1}{n}\mathbb{1} \in \partial V(x(t))$.

By using the same arguments as above, we can prove that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}W(x(t)) < 0$ if $x(t) \notin \mathrm{span}\{\mathbb{1}\}$ and $\tilde{\mathcal{L}}_{\mathcal{F}[h]}W(x(t)) = \{0\}$ if $x(t) \in \mathrm{span}\{\mathbb{1}\}$.

The above analysis implies that all trajectories are bounded. Indeed for any $N \in \mathbb{R}_{+}$ the set $S_N = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant N\}$ is strongly invariant. By Theorem 2 in [Cortés, 2008], the conclusion follows. ∎

Remark 8.15 A stronger assumption is to assume that $g_{ij}(y) = -g_{ji}(-y)$ for all $e_{ij} \in E$ and all $y \in \mathbb{R}$. This would imply that $\sum_{i\in\mathcal{I}} h_i(x) = 0$ for any $x \in \mathbb{R}^n$. Hence for any $x \in \mathbb{R}^n$ and for any $\nu \in \mathcal{F}[h](x)$, we have $\mathbb{1}^\mathsf{T}\nu = 0$. Then any Filippov solution $x(t)$ of system (8.24) satisfies $\mathbb{1}^\mathsf{T}\dot{x}(t) = 0$. Under the same assumption as in Theorem 8.14, the trajectories of (8.24) converge to a consensus value defined by the average of the initial condition.

For the rest of this section, we consider *directed* graphs. In this case, Assumption 8.12 is not sufficient to guarantee convergence to consensus as shown by the following example.

Example 8.16 Consider system (8.24) on the directed graph as in Fig. 8.6b, where the functions g_{ij} are the signum function. Hence the dynamics can be written as

$$\begin{aligned}
\dot{x}_1 &= \mathrm{sign}(x_3 - x_1)\\
\dot{x}_2 &= \mathrm{sign}(x_1 - x_2)\\
\dot{x}_3 &= \mathrm{sign}(x_2 - x_3).
\end{aligned}$$

Suppose that at time t_0, the state satisfies $x(t_0) \in \text{span}\{\mathbb{1}\}$. Then the Filippov set-valued map $\mathcal{F}[h](x(t_0))$ is the same as in (8.14). Hence, by the same argument as in Example 8.5, there are Filippov solutions that exhibit sliding consensus.

For digraphs, we quote the following result for the case that the functions g_{ij} are continuous.

Theorem 8.17 (Papachristodoulou et al. [2010]) *Consider the system (8.24) with continuous functions g_{ij}. If the underlying graph $G = \{V, E\}$ contains a directed spanning tree, then the trajectories of (8.24) achieve consensus asymptotically.*

Extension of Theorem 8.17 to the case of discontinuous functions g_{ij} is a topic for further research.

8.3.4 *Combining node and edge nonlinearities*

The multi-agent system given in (8.7) can be seen as a combination of system (8.12) and system (8.24). For this system, we have the following result.

Theorem 8.18 *Consider system (8.7) defined on a digraph $G = \{V, E\}$, with continuous functions g_{ij}. If one of the following three conditions holds, i.e.,*

(i) $\mathcal{I}^r \cap \mathcal{I}^c$ *is not empty,*

(ii) $|\mathcal{I}^r| = 1$,

(iii) $|\mathcal{I}^r| = 2$, $f_i(0^-)$ *and* $f_i(0^+)$ *exist, and* $f_i(0^-) = -f_i(0^+)$ *for* $i \in \mathcal{I}^r$,

then all Filippov solutions of system (8.12) achieve consensus asymptotically, for all initial conditions.

Proof. Since the proof is similar to the proof of Theorem 8.7, we only provide a sketch of the proof. Recall that $\alpha(t) = \{i \in \mathcal{I} \mid x_i(t) = V(x(t))\}$ and $\beta(t) = \{i \in \mathcal{I} \mid x_i(t) = -W(x(t))\}$.

Let V and W be candidate Lyapunov functions. We will show that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V \leqslant 0$ by considering two cases: $\mathcal{I}^r \nsubseteq \alpha(t)$ and $\mathcal{I}^r \subseteq \alpha(t)$.

When $\mathcal{I}^r \nsubseteq \alpha(t)$, there exists at least one $k \in \alpha(t)$ satisfying $\sum_{j=1}^n a_{kj} g_{kj}(x_j - x_k) < 0$. This implies that the kth component of $\mathcal{F}[h](x(t))$ is contained in $\mathbb{R}_-$. Hence, $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V < 0$.

When $\mathcal{I}^r \subseteq \alpha(t)$, we can use similar arguments as in the proof of Theorem 8.7 to see that the set-valued Lie derivative $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$ is either $\{0\}$ or $\varnothing$ if one of the conditions *(i)*, *(ii)* and *(iii)* holds. Hence $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) \leqslant 0$.

Similarly, we have that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) < 0$ if $\mathcal{I}^r \nsubseteq \beta(t)$, and $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) \leqslant 0$ if $\mathcal{I}^r \subseteq \beta(t)$. Based on Theorem 2 in [Cortés, 2008], the conclusion follows. ■

8.4 CONCLUSIONS

In this chapter, we considered a general model of distributed multi-agent systems defined on a directed graph, with nonlinear discontinuous functions defined on the nodes and edges. Since the right-hand sides of the differential equations are discontinuous, we interpret the solutions in the Filippov sense. Under the crucial assumptions of (i) the graph containing a directed spanning tree, (ii) all nonlinear functions to be sign-preserving, we provided sufficient conditions for all Filippov solutions of a nonlinear consensus protocol to achieve consensus.

9 CONCLUSIONS AND FUTURE RESEARCH

In this thesis, we studied two types of nonsmooth dynamical systems; linear multi-modal systems and multi-agent systems with a nonlinear communication protocol. First, we studied two problems for linear multi-modal systems: the disturbance decoupling problem and the fault detection and isolation problem. We used ideas and results from geometric control theory for linear systems and nonsmooth dynamical systems, but had to take state-dependent switching into account. Finally, we studied multi-agent systems with nonlinear consensus protocols. The nonlinear, possibly discontinuous, functions appear on the nodes and edges of the directed communication graph.

9.1 DISTURBANCE DECOUPLING FOR LINEAR MULTI-MODAL SYSTEMS

In Chapter 2 we started by studying the disturbance decoupling problem for a simple class of piecewise linear systems: bimodal systems. We presented necessary and sufficient conditions for such systems to be disturbance decoupled. Furthermore, we have given a complete characterization of the solvability of the disturbance decoupling problem with mode-independent and mode-dependent feedback controllers. In order to verify the presented conditions, we provided subspace algorithms.

Next, in Chapter 3, we directed our attention to more general continuous piecewise affine systems. We established necessary conditions as well as sufficient conditions for a piecewise affine system to be disturbance decoupled. Unfortunately, these conditions do not coincide in general. However, we identified a number of particular cases for which they do coincide. Furthermore, we provided conditions for the existence of a mode-independent static feedback controller that renders a given piecewise affine system disturbance decoupled. All presented conditions are geometric in nature and easily verifiable.

Then in Chapter 4, we turned our attention to linear complementarity systems of index zero, which can be rewritten as a particular class of linear multi-modal systems. The resulting system is closely related to the piecewise affine systems in

Chapter 3, and its linear subsystems exhibit a certain geometric structure. By exploiting this geometric structure, we provided necessary and sufficient conditions for a linear complementarity system to be disturbance decoupled. These conditions are more compact and perhaps more insightful than the conditions that we found for general piecewise affine systems in Chapter 3.

Finally, in Chapter 5, we combined the ideas developed in the previous chapters to tackle the disturbance decoupling problem for a general class of linear multi-modal systems. We presented necessary conditions and sufficient conditions, geometric in nature, under which such multi-modal systems are disturbance decoupled. These results generalize the existing results in the literature on switched linear systems [Yurtseven et al., 2012], as well as bimodal systems (Chapter 2), conewise linear systems (Chapter 3) and linear complementarity systems namely those of index zero (Chapter 4). In addition, these results led to novel necessary and sufficient conditions for passive-like linear complementarity systems whose disturbance decoupling properties were not been studied before. For the presented general linear multi-modal system the necessary condition and the sufficient condition for being disturbance decoupled do not coincide. However, we presented several conditions under which these conditions do coincide.

In Chapter 4 and Chapter 5 we studied under what conditions a linear complementarity system or a general linear multi-modal system is disturbance decoupled; rendering such systems disturbance decoupled by means of feedback is the next step. Finding a static state feedback such that the resulting closed-loop system satisfies the necessary condition for disturbance decoupledness becomes a linear algebraic problem and can be solved mimicking the footsteps for the linear case. Studying the disturbance decoupling problem under different feedback schemes is a future research possibility. Regarding the disturbance decoupling problem for continuous piecewise affine systems, further research possibilities include investigating the gap between the necessary conditions and sufficient conditions as well as studying mode-dependent state feedback. Another possible future research line is to study the extension to Filippov solutions. Furthermore, the results for the linear complementarity systems might be extended to the more general case with a not necessarily symmetric but positive semi-definite matrix H for which there exists a positive symmetric matrix K such that $KGu = N^T u$ for all $u \in \ker(H + H^T)$.

9.2 FAULT DETECTION AND ISOLATION

The next geometric control problem we addressed for bimodal piecewise linear systems is the fault detection and isolation problem. We used a geometric approach to find sufficient conditions for the existence of an observer that produces residuals that are sufficiently informative about the fault. A method for finding an asymptotic observer based on bilinear matrix inequalities has been discussed. A possible future research direction includes extending these ideas and results towards multi-modal piecewise linear systems.

As a by-product, in Chapter 7 we considered the fault detection and isolation problem for a class of linear systems defined on an undirected graph, containing faultable vertices and observer vertices. We provided a characterization of the so-called conditioned invariant subspaces of such systems that satisfy the distance-information preservation property. These subspaces play a major role in the analysis of fault detection as well as in the design of fault detectors. Based on this characterization, we presented a graph-topological sufficient condition for the so-called output separability requirement, which is the crux of the fault detection problem in the setting of geometric control. Furthermore, we have presented a condition under which the output separability fails for the class of distance-information preserving matrices. Future research problems include investigating sharper sufficient conditions, devising observer vertex selection methods and formulating conditions that would guarantee the asymptotic stability of fault detectors.

9.3 CONSENSUS DYNAMICS WITH NONLINEARITIES

The last problem that we studied in this thesis is a consensus problem for a general model of distributed multi-agent systems defined on a directed graph, with nonlinear discontinuous functions defined on the nodes and edges. In Chapter 8, we provided sufficient conditions for all Filippov solutions of a nonlinear consensus protocol to achieve consensus, whenever the underlying communication graph contains a directed spanning tree and all the nonlinear functions are sign-preserving. Future research possibilities include extending the results to multi-agent systems with switching communication topologies and studying higher-order nonlinear consensus protocols.

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SUMMARY

In this thesis, we used ideas and results from control theory to study several problems for two types of nonsmooth dynamical systems; linear multi-modal systems and multi-agent systems with a nonlinear communication protocol.

The first problem that we addressed is the disturbance decoupling problem for linear multi-modal systems. We started by studying the disturbance decoupling problem for a simple class of piecewise linear systems: continuous bimodal systems. We presented necessary and sufficient conditions for such systems to be disturbance decoupled. Furthermore, we have given a complete characterization of the solvability of the disturbance decoupling problem with both state feedback controllers and dynamic feedback controllers.

Thereupon we directed our attention to more general continuous piecewise affine systems. We established necessary conditions as well as sufficient conditions for a piecewise affine system to be disturbance decoupled. Furthermore, we identified a number of particular cases for which these conditions coincide. Additionally, we provided conditions for the existence of a mode-independent static feedback controller that renders a given piecewise affine system disturbance decoupled.

After that we studied the disturbance decoupling problem for linear complementarity systems of index zero. Such systems can be rewritten as a particular class of linear multi-modal systems. We exploited the geometric structure that the resulting linear subsystems exhibit and provided necessary and sufficient conditions for a linear complementarity system to be disturbance decoupled.

Finally, we tackle the disturbance decoupling problem for a general class of linear multi-modal systems. We presented necessary conditions and sufficient conditions under which such multi-modal systems are disturbance decoupled. Furthermore, we presented several conditions under which these conditions coincide. All presented conditions are geometric in nature and easily verifiable. These results generalize almost all results in the first chapters of this thesis as well as the existing literature on switched linear systems. In addition, these results led to novel necessary and sufficient conditions for passive-like linear complementarity systems.

The second geometric control problem we addressed is the fault detection and isolation problem for bimodal piecewise linear systems. Using a geometric approach, we presented sufficient conditions for the existence of an observer that produces residuals that are sufficiently informative about the fault. In addition, we discussed a method for finding an asymptotic observer based on bilinear matrix inequalities.

As a by-product, we studied the fault detection and isolation problem for a class of linear systems defined on an undirected graph, containing faultable vertices and observer vertices. In this graph, two disjoint sets of agents are identified in the network: those prone to failure and those whose output is measurable. For systems that satisfy the distance-information preservation property we provided a characterization of the so-called conditioned invariant subspaces. Based on this characterization, we presented a graph-theoretical sufficient condition for the so-called output separability requirement, which is the crux of the fault detection problem in the setting of geometric control.

The last problem that we have studied in this thesis is a consensus problem for multi-agent systems defined on a directed graph, with nonlinear discontinuous functions defined on the nodes and edges. We provided sufficient conditions for all Filippov solutions of a nonlinear consensus protocol to achieve consensus, whenever the underlying communication graph contains a directed spanning tree and all the nonlinear functions are sign-preserving.

SAMENVATTING

In dit proefschrift maken we gebruik van ideeën en resultaten uit de systeem- en regeltheorie om verschillende problemen te bestuderen voor twee soorten niet-gladde dynamische systemen, namelijk lineaire multimodale systemen en multi-agent systemen met een niet-lineair communicatieprotocol.

Het eerste probleem waar we ons op richten is het probleem van storingsontkoppeling voor lineaire multimodale systemen. Een systeem is storingsontkoppeld als het uitgangssignaal van het systeem onafhankelijk is van eventuele storingsingangen. We bestuderen het storingsontkoppelingsprobleem eerst voor een eenvoudige klasse van stuksgewijs lineaire systemen, namelijk bimodale systemen. In Hoofdstuk 2 presenteren we noodzakelijke en voldoende voorwaarden voor deze systemen om storingsontkoppeld te zijn. Verder geven we een volledige karakterisering van de oplosbaarheid van het storingsontkoppelingsprobleem met behulp van toestandsterugkoppeling en dynamische uitgangsterugkoppeling.

Daarna richten we ons op algemenere stukgewijs affiene systemen. In dergelijke systemen is de toestandsruimte opgedeeld in polyhedrale deelgebieden, elk met een corresponderend affien dynamisch systeem. Welk affien systeem actief is, hangt af van het deelgebied waarin de toestand zich op dat moment bevindt. In Hoofdstuk 3 geven we noodzakelijke voorwaarden evenals voldoende voorwaarden voor storingsontkoppeling van een stuksgewijs affien systeem. Bovendien identificeren we een aantal bijzondere gevallen waarin deze voorwaarden samenvallen. Daarnaast geven we voorwaarden voor het bestaan van een mode-onafhankelijke toestandsterugkoppeling die een gegeven stuksgewijs affien systeem storingsontkoppeld maakt.

Vervolgens onderzoeken we in Hoofdstuk 4 het storingsontkoppelingprobleem voor lineaire complementariteitssystemen met index nul. Deze systemen kunnen worden herschreven als lineaire multimodale systemen. We maken gebruik van de geometrische structuur van de resulterende lineaire subsystemen om noodzakelijke en voldoende voorwaarden te vinden voor de storingsontkoppeling van deze lineaire complementariteitssystemen.

Tot slot bestuderen we het storingsontkoppelingsprobleem voor een algemene klasse van lineaire multimodale systemen in Hoofdstuk 5. We presenteren noodzakelijke voorwaarden en voldoende voorwaarden voor de storingsontkoppeling van dergelijke multimodale systemen. Verder beschrijven we een aantal speciale gevallen waarin deze voorwaarden samenvallen. Alle gepresenteerde voorwaarden zijn meetkundig van aard en gemakkelijk te verifiëren. Deze resultaten generaliseren vrijwel alle resultaten in de eerste hoofdstukken van dit proefschrift en ook die in de bestaande literatuur over geschakelde lineaire systemen. Daarnaast gebruiken we deze resultaten om nieuwe noodzakelijke en voldoende voorwaarden af te leiden voor storingsontkoppeling van een ander type lineaire complementariteitssystemen, namelijk die systemen die passief gemaakt kunnen worden met pole-shifting.

Het tweede probleem waar we ons op richten is het probleem van foutdetectie en -isolatie voor bimodale stuksgewijs lineaire systemen. Gebruikmakend van de meetkundige aanpak binnen de lineaire systeemtheorie, presenteren we voldoende voorwaarden voor het bestaan van een op een waarnemer gebaseerd foutdetectiefilter. Daarnaast bespreken we een methode voor het vinden van een asymptotische waarnemer gebaseerd op bilineaire matrixongelijkheden.

Vervolgens bestuderen we foutdetectie en -isolatie voor een klasse van lineaire multi-agent systemen. Zulke systemen zijn gedefinieerd op een ongerichte graaf, waarbij elke knoop een agent representeert. In deze graaf wijzen we twee disjuncte deelverzamelingen van agents aan: de storingsgevoelige agents en de waarnemende agents. Het doel is om uit de uitgangssignalen van de waarnemende agents af te leiden of de storingsgevoelige agents aan storing onderhevig zijn, en zo ja, welke agents dat dan zijn. Voor afstandsinformatiebehoudende systemen geven we een karakterisering van de zogenaamde conditioneel invariante deelruimten. Gebaseerd op deze karakterisering vinden we vervolgens een graaftheoretische voldoende voorwaarde voor foutdetectie en -isolatie.

Het laatste probleem dat we bestuderen in dit proefschrift is een consensusprobleem voor multi-agent systemen, waarbij de communicatiegraaf gericht is en er bovendien niet-lineaire discontinue functies voorkomen op de knopen of kanten. In het geval dat de onderliggende communicatiegraaf een gerichte opspannende boom bevat en alle niet-lineaire functies tekenbehoudend zijn, geven we voldoende voorwaarden waaronder alle Filippov oplossingen van een dergelijk niet-lineaire consensusprotocol consensus bereiken.

BIOGRAPHY

Annerosa Roelienke Fleur Everts was born on 8 June 1988 in Hoogeveen, The Netherlands. She obtained both her bachelor's degree in Mathematics and her master's degree in Algebra and Geometry with the designation cum laude from the University of Groningen. After that she started her PhD in the research group Systems, Control and Applied Analysis (SCAA) at the University of Groningen, under the supervision of M. Kanat Çamlıbel. Her research interests include systems and control theory, geometric control theory, piecewise affine dynamical systems, hybrid systems, multi-agent networks, and graph theory.

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Anneroos *Groningen, 24 October 2016*

COLOPHON

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