

When is a linear multi-modal system disturbance decoupled?

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Abstract

In this paper we study the question under which conditions a linear multi-modal system is disturbance decoupled. We establish necessary and sufficient geometric conditions from which the existing results on switched linear systems and conewise linear systems can be recovered as special cases. Also, we apply these conditions to a class of linear complementarity systems in order to obtain a more crisp characterization.

Keywords: Disturbance decoupling, hybrid systems, piecewise-affine systems, conewise linear systems, switched systems, linear complementarity systems.

1. Introduction

Annihilating or reducing the effects of disturbances is of major importance virtually in every real-life control problem. Designing feedback laws that decouple the disturbances from a certain to-be-controlled output constitute the well-known disturbance decoupling problem. The study of this problem for linear systems led to the development of geometric control theory [2, 3, 38] which provided solutions to numerous control problems as well as a deep understanding of the dynamics of linear systems [37, 4, 29] and (smooth) nonlinear systems [23, 22].

Geometric approach to linear systems is among many fields on which Jan Willems made a major impact. Examples are his seminal work on almost invariant subspaces [33, 34], (almost) disturbance decoupling [35, 32], and application of geometric ideas to singular optimal control problems [36].

In this paper, we focus on a class of hybrid dynamical systems and provide necessary and sufficient geometric conditions under which the disturbances are decoupled from pre-specified to-be-controlled outputs. Within the hybrid systems, the results on disturbance decoupling problem are limited to jumping hybrid systems [12], continuous piecewise affine systems [14], a class of linear

complementarity systems [16], and switched linear systems [11, 24, 25, 26, 39, 40]. The results presented in these papers very much resemble those for the linear systems although their derivation is much harder, in particular, in the presence of state-dependent switching [14, 16].

Within this paper, we first lay a general framework that contains a particular class of switched linear systems, linear complementarity systems, and the so-called conewise linear systems (as well as combinations of these) as particular cases. Later, we investigate necessary and sufficient conditions for disturbances to be decoupled within this general framework. In addition, we show that all the existing results for the hybrid systems mentioned above can be recovered from the presented results as special cases.

Furthermore, we study a class of linear complementarity systems in detail in order to find novel necessary and sufficient conditions for this kind of systems to be disturbance decoupled.

The organization of the paper is as follows. We start with some preliminaries and notation in Section 2. We introduce general linear multi-modal systems in Section 3 and discuss a few special cases. In Section 4 we give the definition of the property of being disturbance decoupled for a linear multi-modal system, and present our main results. In Theorem 8 we give a necessary condition and in Theorem 9 a sufficient condition for a linear multi-modal system to be disturbance decoupled. In

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Corollary 11 we show that in some cases the necessary condition and the sufficient condition coincide. We apply these results in Section 5 to the special cases that we introduced in Section 3. For one type of linear complementarity systems, this will lead to novel results, stated in Theorem 14. The paper closes with the conclusions and discussions of possible future work in Section 6.

2. Preliminaries and notation

For a vector v we write n_v as its dimension. For two vectors v and w , we let $\text{col}(v, w)$ denote the column vector that is obtained by stacking v and w . A cone is a subset of a vector space that is closed under multiplication by positive scalars. The relative interior of a set $S \subseteq \mathbb{R}^n$ is defined as

$$\text{rint}(S) := \{x \in S : \exists \epsilon > 0, N_\epsilon(x) \cap \text{aff}(S) \subseteq S\},$$

where $N_\epsilon(x)$ is an ϵ -neighborhood of x and $\text{aff}(S)$ is the affine hull of S .

Consider the linear system $\Sigma = \Sigma(A, B, C, D)$ given by

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1a)$$

$$y(t) = Cx(t) + Du(t) \quad (1b)$$

where the input u , state x , and output y have dimensions n_u , n_x and n_y respectively. In what follows, we quickly introduce some of the fundamental notions of geometric control theory of linear systems for the sake of completeness. We refer to [16] for more details.

The *controllable subspace* of Σ is the smallest A -invariant subspace containing $\text{im } B$. We will denote it by

$$\langle A \mid \text{im } B \rangle := \text{im } B + A \text{im } B + \dots + A^{n_x-1} \text{im } B. \quad (2)$$

A subspace $\mathcal{T} \subseteq \mathbb{R}^n$ is called an *input containing conditioned invariant subspace* of Σ if

$$\begin{bmatrix} A & B \end{bmatrix} ((\mathcal{T} \times \mathbb{R}^{n_u}) \cap \ker \begin{bmatrix} C & D \end{bmatrix}) \subseteq \mathcal{T}.$$

It is well-known that a subspace \mathcal{T} is an input containing conditioned invariant subspace if and only if there exists a matrix $L \in \mathbb{R}^{n \times p}$ such that

$$(A + LC)\mathcal{T} \subseteq \mathcal{T} \quad \text{and} \quad \text{im}(B + LD) \subseteq \mathcal{T}. \quad (3)$$

The *strongly reachable subspace* of Σ is the smallest (with respect to the subspace inclusion) input

containing conditioned invariant subspace and will be denoted by $\mathcal{T}^*(\Sigma)$.

It follows from (3) with the choice of $L = 0$ that the controllable subspace is an input containing conditioned invariant subspace. Hence, we have

$$\mathcal{T}^*(\Sigma) \subseteq \langle A \mid \text{im } B \rangle. \quad (4)$$

Let K and L be $m \times n$ and $n \times p$ matrices, respectively. Also let $\Sigma_{K,L}$ denote the system $\Sigma(A + BK + LC + LDK, B + LD, C + DK, D)$. It can easily be verified that

$$\mathcal{T}^*(\Sigma_{K,L}) = \mathcal{T}^*(\Sigma). \quad (5)$$

A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called an *output nulling controlled invariant subspace* of Σ if

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V} \times \{0\}) + \text{im} \begin{bmatrix} B \\ D \end{bmatrix}.$$

The *weakly unobservable subspace* of Σ is the largest (with respect to the subspace inclusion) output nulling controlled invariant subspace and will be denoted by $\mathcal{V}^*(\Sigma)$.

It is well-known that the transfer matrix $D + C(sI - A)^{-1}B$ is *right invertible* as a rational matrix if and only if

$$\mathcal{V}^*(\Sigma) + \mathcal{T}^*(\Sigma) = \mathbb{R}^{n_x} \quad \text{and} \quad \text{rank} \begin{bmatrix} C & D \end{bmatrix} = n_y.$$

Straightforward linear algebra arguments show that these conditions are equivalent to

$$\text{im } D + C\mathcal{T}^*(\Sigma) = \mathbb{R}^{n_y}. \quad (6)$$

3. Linear multi-modal systems

In this paper we consider linear multi-modal systems given by the differential inclusion

$$\dot{x}(t) \in Ax(t) + Ed(t) + \Phi(y(t)) \quad (7a)$$

$$y(t) = Cx(t) + Fd(t) \quad (7b)$$

$$z(t) = Jx(t) \quad (7c)$$

where x is the state, d is the disturbance, y is the selection output, z is the to-be-controlled output, A, C, E, F and J are matrices of appropriate sizes and $\Phi : \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_x}$ is a set-valued map satisfying

$$\Phi(y) = \{M_i y \mid i \in \mathcal{I}, y \in \mathcal{Y}_i\},$$

where \mathcal{I} is a finite index set, $\{\mathcal{Y}_i\}_{i \in \mathcal{I}}$ is a collection of cones in \mathbb{R}^{n_y} , and $\{M_i\}_{i \in \mathcal{I}}$ is a collection

of $n_x \times n_y$ matrices. The cones \mathcal{Y}_i are not necessarily *solid* (i.e., n_y -dimensional). Moreover, the cones may overlap and their union does not have to be equal to \mathbb{R}^{n_y} . Without loss of generality we can assume that the matrix $\begin{bmatrix} C & F \end{bmatrix}$ has full row rank.

Let $T > 0$. For a given initial state x_0 and an integrable disturbance d we call an absolutely continuous function $x : [0, T] \rightarrow \mathbb{R}^{n_x}$ a solution on $[0, T]$ of system (7) if (7a) holds for almost all $t \in [0, T]$ and $x(0) = x_0$. If $T = +\infty$, we simply say that x is a (complete) solution of (7). In the sequel, we will allow multiple solutions for a given initial state and disturbance but make two assumptions regarding the existence of solutions.

The first assumption we make is that *local* solutions can be extended to complete solutions.

Assumption 1 If the system (7) admits a local solution x_T on $[0, T)$ for some $T > 0$, initial state x_0 , and disturbance d , then there exists a complete solution x for the same initial state x_0 and disturbance satisfying $x(t) = x_T(t)$ for all $t \in [0, T)$.

The second assumption regarding the existence of solutions requires that the disturbances are not restricted by the dynamics of the system.

Assumption 2 If the system (7) admits a complete solution for some initial state and disturbance, then there exists a complete solution for the same initial state and for any disturbance.

Later on, we will elaborate on these assumptions when we discuss specific classes of systems that fall into the framework of (7).

We say that an initial state is *feasible* if for all locally integrable disturbances d there exists a complete solution of (7). The set of all feasible states will be denoted by \mathcal{X}_0 .

To simplify the notation, we define

$$A_i = A + M_i C, \quad E_i = E + M_i F \quad (8)$$

and rewrite system (7) as

$$\dot{x}(t) \in \{A_i x(t) + E_i d(t) \mid y(t) \in \mathcal{Y}_i\} \quad (9a)$$

$$y(t) = Cx(t) + Fd(t) \quad (9b)$$

$$z(t) = Jx(t) \quad (9c)$$

We will work mainly with this form of the linear multi-modal system in the rest of the paper.

Examples of systems that fall into this framework include switched linear systems, conewise linear systems, and linear complementarity problems, which we discuss next.

Example 3 (Switched Linear Systems) We consider the following particular class of linear switched systems

$$\dot{x}(t) = A_{\sigma(t)}x(t) + E_{\sigma(t)}d(t) \quad (10a)$$

$$z(t) = Jx(t), \quad (10b)$$

where σ is a switching signal from $\mathbb{R}_{\geq 0}$ to a finite index set \mathcal{I} . By taking $A = A_j$ and $E = E_j$ for some $j \in \mathcal{I}$, we can rewrite (10) in the form of a multi-modal system as

$$\dot{x}(t) \in Ax(t) + Ed(t) + \Phi(y) \quad (11a)$$

$$y(t) = \text{col}(x(t), d(t)), \quad (11b)$$

$$z(t) = Jx(t), \quad (11c)$$

where

$$\Phi(y) = \{[A_i - A \quad E_i - E]y \mid i \in \mathcal{I}, y \in \mathcal{Y}_i\}$$

and $\mathcal{Y}_i = \mathbb{R}^{n_x}$ for all i . Note that Assumptions 1 and 2 naturally hold for switched linear systems and $\mathcal{X}_0 = \mathbb{R}^{n_x}$.

Example 4 (Conewise Linear Systems) A *continuous* function $\Phi : \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_x}$ is said to be *conewise linear* if there exist a finite family of solid polyhedral cones $\{\mathcal{Y}_i\}_{i \in \mathcal{I}}$ with $\cup_{i \in \mathcal{I}} \mathcal{Y}_i = \mathbb{R}^{n_y}$ and $n_x \times n_y$ matrices $\{M_i\}_{i \in \mathcal{I}}$ such that $g(y) = M_i y$ for $y \in \mathcal{Y}_i$.

Consider systems of the form

$$\dot{x}(t) = Ax(t) + Ed(t) + \Phi(y(t)) \quad (12a)$$

$$y(t) = Cx(t) + Fd(t) \quad (12b)$$

$$z(t) = Jx(t) \quad (12c)$$

where $\Phi : \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_x}$ is a continuous conewise linear function. These systems will be called *conewise linear systems* (CLS) and were studied in [10, 1, 8]. Naturally, CLSs fall into the framework of (7). As the union of the (solid) cones \mathcal{Y}_i is the entire \mathbb{R}^{n_y} , Assumptions 1 and 2 are satisfied and $\mathcal{X}_0 = \mathbb{R}^{n_x}$.

Example 5 (Complementarity Systems) We consider the linear complementarity system (LCS)

$$\dot{x}(t) = Ax(t) + Ed(t) + G\zeta(t) \quad (13a)$$

$$\eta(t) = Nx(t) + Rd(t) + H\zeta(t) \quad (13b)$$

$$0 \leq \zeta(t) \perp \eta(t) \geq 0 \quad (13c)$$

$$z(t) = Jx(t), \quad (13d)$$

where $\zeta, \eta \in \mathbb{R}^{n_\eta}$ are the so-called complementarity variables and all involved matrices are of appropriate dimensions.

Here, the inequalities for vectors are component-wise inequalities and \perp denotes orthogonality. Linear complementarity systems are encountered in applications from various areas of engineering as well as operations research [30, 18, 28, 31]. For the work on the analysis and control of linear complementarity systems, we refer to [19, 20, 7, 6, 5, 17, 21, 9].

In this paper, we focus on two particular classes of linear complementarity systems that were heavily studied in the literature:

1. H is a P -matrix, that is a matrix whose principal minors are all positive (see e.g. [13]).
2. $R = 0$, $H = 0$, and NG is a symmetric positive definite matrix.

In what follows we will briefly derive the corresponding linear multi-modal systems for these two cases by skipping technical details for which we refer to [19] for the first case and [9] for the second.

In the case that H is a P -matrix, the LCS (13) boils down to the multi-modal system

$$\dot{x}(t) \in \{A_\alpha x(t) + E_\alpha d(t) \mid \alpha \in \mathcal{I}, y(t) \in \mathcal{Y}_\alpha\} \quad (14a)$$

$$y(t) = Cx(t) + Fd(t) \quad (14b)$$

$$z(t) = Jx(t) \quad (14c)$$

where \mathcal{I} is the set of all subsets of $\{1, 2, \dots, n_\eta\}$, and

$$C = N$$

$$F = R$$

$$A_\alpha = A - G_{\bullet\alpha}(H_{\alpha\alpha})^{-1}N_{\alpha\bullet}$$

$$E_\alpha = E - G_{\bullet\alpha}(H_{\alpha\alpha})^{-1}N_{\alpha\bullet}$$

$$\mathcal{Y}_\alpha = \{y \in \mathbb{R}^{n_\eta} \mid \begin{bmatrix} -(H_{\alpha\alpha})^{-1}I_{\alpha\bullet} \\ I_{\alpha^c\bullet} - H_{\alpha^c\alpha}(H_{\alpha\alpha})^{-1}I_{\alpha\bullet} \end{bmatrix} y \geq 0\}.$$

Note that in the above on the left-hand side the subscript α is used as an index, whereas on the right-hand side the subscript $\alpha\beta$ selects rows α and columns β of a matrix, for given index sets α and β . Here, the \bullet means selecting all rows or columns and α^c denotes the complement of α in $\{1, 2, \dots, n_\eta\}$.

In the case that $R = 0$, $H = 0$, and NG is a symmetric positive definite matrix, the LCS (13) boils down to the multi-modal system (14) where

where $\alpha \subseteq \{1, 2, \dots, n_\eta\}$ and

$$C = \begin{bmatrix} N \\ NA \end{bmatrix}$$

$$F = \begin{bmatrix} 0 \\ NE \end{bmatrix}$$

$$A_\alpha = A - G_{\bullet\alpha}(N_{\alpha\bullet}G_{\bullet\alpha})^{-1}N_{\alpha\bullet}A$$

$$E_\alpha = E - G_{\bullet\alpha}(N_{\alpha\bullet}G_{\bullet\alpha})^{-1}N_{\alpha\bullet}E$$

$$\mathcal{Y}_\alpha = \{y \in \mathbb{R}^{2n_\eta} \mid \begin{bmatrix} I_{\alpha^c\bullet} & 0 \\ 0 & -(N_{\alpha\bullet}G_{\bullet\alpha})^{-1}I_{\alpha\bullet} \end{bmatrix} y \geq 0, \\ [I_{\alpha\bullet} \quad 0] y = 0\}.$$

For both cases, Assumptions 1 and 2 are satisfied [19, 9]. We have $\mathcal{X}_0 = \mathbb{R}^{n_x}$ (see e.g. [19]) in case H is a P -matrix and $\mathcal{X}_0 = \{x_0 \mid Cx_0 \geq 0\}$ for the second case (see e.g. [9]).

4. Disturbance decoupled systems

We start with the following definition of a disturbance decoupled system.

Definition 6 We say that system (9) is *disturbance decoupled* if for any given feasible initial state $x_0 \in \mathcal{X}_0$ and any two solutions $(x_1(t), y_1(t), z_1(t))$ and $(x_2(t), y_2(t), z_2(t))$, corresponding to any two locally integrable disturbances $d_1(t)$ and $d_2(t)$ respectively, satisfy

$$z_1(t) = z_2(t)$$

for all $t \geq 0$.

In this paper we investigate when system (9) is disturbance decoupled.

Throughout the paper we assume the following.

Assumption 7 For each $i \in \mathcal{I}$, the subspace $\text{im } F + C\langle A_i \mid \text{im } E_i \rangle$ and the cone \mathcal{Y}_i satisfy

- i. $\text{im } F + C\langle A_i \mid \text{im } E_i \rangle \subseteq \text{span}(\mathcal{Y}_i)$,
- ii. $(\text{im } F + C\langle A_i \mid \text{im } E_i \rangle) \cap \text{rint}(\mathcal{Y}_i) \neq \emptyset$, or \mathcal{Y}_i is solid.

The first assumption is trivial when each cone \mathcal{Y}_i is solid. A consequence of this assumption is that $\text{im } F \subseteq \bigcap_{i \in \mathcal{I}} \text{span}(\mathcal{Y}_i)$. The second assumption assures a certain ‘liveliness’ of each cone \mathcal{Y}_i ; for every cone \mathcal{Y}_i there exists a point x_0 and a locally integrable disturbance $d(t)$ such that $y^{x_0, d}(t)$ stays in $\text{rint}(\mathcal{Y}_i)$ for some time t . If $F + C(sI - A)^{-1}E$

right invertible, using (6), (5), and (4), one can see that $\text{im } F + C\langle A_i \mid \text{im } E_i \rangle = \mathbb{R}^{n_u}$, which implies the second assumption.

A necessary condition for a linear multi-modal system to be disturbance decoupled is stated in the following theorem.

Theorem 8 *If a linear multi-modal system of the form (9), satisfying Assumptions 1, 2 and 7, is disturbance decoupled, then*

$$\sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle \subseteq \ker J. \quad (15)$$

Proof. Fix $i \in \mathcal{I}$. Since $\begin{bmatrix} C & F \end{bmatrix}$ is of full row rank, there exist an $x_0 \in \mathbb{R}^{n_x}$ and a $d \in \mathbb{R}^{n_d}$ such that

$$y_0 := Cx_0 + Fd \in \text{rint}(\mathcal{Y}_i).$$

If the first condition in Assumption 7 holds, then we can even pick $x_0 \in \langle A_i \mid \text{im } E_i \rangle$. Consider the solution $(\tilde{x}(t), \tilde{y}(t))$ of the following linear system

$$\dot{\tilde{x}}(t) = A_i \tilde{x}(t) + E_i d(t) \quad (16a)$$

$$\tilde{y}(t) = C\tilde{x}(t) + Fd(t), \quad (16b)$$

where $d(t) = d$ and with $\tilde{x}(0) = x_0$. If \mathcal{Y}_i is solid, then the continuity of $\tilde{y}(t)$ implies that there exists an $\varepsilon > 0$ such that $\tilde{y}(t) \in \text{rint}(\mathcal{Y}_i)$ for all $t \in [0, \varepsilon]$. In the case that we have $x_0 \in \langle A_i \mid \text{im } E_i \rangle$, we see that $\tilde{x}(t) \in \langle A_i \mid \text{im } E_i \rangle$ for all $t \geq 0$. Hence, by Assumption 7(i),

$$\tilde{y}(t) \in \text{im } F + C\langle A_i \mid \text{im } E_i \rangle \subseteq \text{span}(\mathcal{Y}_i)$$

for all $t \geq 0$. Since $\tilde{y}(t)$ is continuous, and $\tilde{y}(0) = y_0 \in \text{rint}(\mathcal{Y}_i)$, it follows that there again exists an $\varepsilon > 0$ such that $\tilde{y}(t) \in \text{rint}(\mathcal{Y}_i)$ for all $t \in [0, \varepsilon]$.

Let e be any vector in \mathbb{R}^{n_d} , then we have

$$Cx_0 + F(d + \mu e) = y_0 + F\mu e \in \text{span}(\mathcal{Y}_i),$$

for any $\mu \in \mathbb{R}$, since $\text{im } F \subseteq \text{span}(\mathcal{Y}_i)$ by Assumption 7(i). By taking $|\mu|$ sufficiently small, we have $Cx_0 + F(d + \mu e) \in \text{rint}(\mathcal{Y}_i)$. Let $\tilde{x}_e(t)$ be the solution of (16) for the constant disturbance $d_e(t) = d + \mu e$ and initial condition $\tilde{x}_e(0) = x_0$, with corresponding output $\tilde{y}_e(t)$. For $\tilde{y}_e(t)$ there is an $\varepsilon_e > 0$ such that $\tilde{y}_e(t) \in \text{rint}(\mathcal{Y}_i)$ for $t \in [0, \varepsilon_e]$.

Let $\varepsilon^* = \min(\varepsilon, \varepsilon_e)$. Due to Assumption 1, we can extend $\tilde{x}(t)$ and $\tilde{x}_e(t)$ from $t = \varepsilon^*$ onwards to obtain complete solutions $x(t)$ and $x_e(t)$ of system (9), with corresponding outputs $(y(t), z(t))$ and

$(y_e(t), z_e(t))$, respectively. Moreover, $x_0 \in \mathcal{X}_0$ due to Assumption 2.

Since system (9) is disturbance decoupled, we have that

$$z(t) - z_e(t) = J(x(t) - x_e(t)) = 0 \quad (17)$$

for all $t \geq 0$. Since $d(t)$ and $d_e(t)$ are constant, we can differentiate (17) repeatedly and evaluate at $t = 0$ to obtain

$$JA_i^k E_i \mu e = 0$$

for all $k \geq 0$. Since this holds for all $e \in \mathbb{R}^{n_d}$, we have $JA_i^k E_i = 0$ for all k . Consequently, by (2), we have $\langle A_i \mid \text{im } E_i \rangle \subseteq \ker J$. As this holds for all $i \in \mathcal{I}$, we can conclude that (15) holds. ■

Next we give a sufficient condition for a linear multi-modal system to be disturbance decoupled. For this purpose, we define the subspaces

$$\mathcal{A} := \sum_{i,j \in \mathcal{I}} \text{im}(A_j - A_i), \quad \mathcal{E} := \sum_i \text{im } E_i. \quad (18)$$

Theorem 9 *If there is a subspace $\mathcal{V} \subseteq \ker J$ such that $A_i \mathcal{V} \subseteq \mathcal{V}$ for each $i \in \mathcal{I}$, $\mathcal{E} \subseteq \mathcal{V}$ and $\mathcal{A} \subseteq \mathcal{V}$, then the linear multi-modal system (9) satisfying Assumption 7 is disturbance decoupled.*

Proof. Let $x_0 \in \mathcal{X}_0$ be any given feasible initial state, and let $n_v = \dim \mathcal{V}$. Furthermore, let $\{\xi_1, \xi_2, \dots, \xi_{n_x}\}$ be a basis for \mathbb{R}^{n_x} such that $\{\xi_1, \xi_2, \dots, \xi_{n_v}\}$ forms a basis for \mathcal{V} . With respect to these coordinates, we can write every $x \in \mathbb{R}^{n_x}$ uniquely as $x = \text{col}(v, w)$ for some $v \in \mathbb{R}^{n_v}$ and $w \in \mathbb{R}^{n_x - n_v}$ such that $\text{col}(v, 0) \in \mathcal{V}$. Since \mathcal{V} is A_i -invariant for each $i \in \mathcal{I}$ and $\mathcal{E} \subseteq \mathcal{V} \subseteq \ker J$, with respect to the new coordinates we have

$$A_i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ 0 & A_{22}^i \end{bmatrix}, \quad E_i = \begin{bmatrix} E_1^i \\ 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & J_2 \end{bmatrix},$$

for every $i \in \mathcal{I}$, where $A_{11}^i \in \mathbb{R}^{n_v \times n_v}$, $E_1^i \in \mathbb{R}^{n_v \times n_d}$ and $J_2 \in \mathbb{R}^{(n_x - n_v) \times n_x}$. Let $x_0 = \text{col}(v_0, w_0)$, and let $d(t)$ be any locally integrable disturbance. Write $x(t) = \text{col}(v(t), w(t))$, then $v(t)$ and $w(t)$ satisfy

$$\dot{v}(t) \in \{A_{11}^i v(t) + A_{12}^i w(t) + E_1^i d(t) \mid \text{for } i \in \mathcal{I} \text{ s.t. } C \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} + Fd(t) \in \mathcal{Y}_i\}$$

$$\dot{w}(t) = A_{22}^i w(t)$$

$$z(t) = J_2 w(t)$$

for almost all t , with $v(0) = v_0$ and $w(0) = w_0$. Since $\text{im}(A_j - A_i) \subseteq \mathcal{V}$ for all $i, j \in \mathcal{I}$, we have $A_{22}^i = A_{22}^j$ for all $i, j \in \mathcal{I}$. Therefore, $w(t)$ will satisfy the linear differential equation

$$\dot{w}(t) = A_{22}^i w(t), \quad w(0) = w_0$$

for any fixed $i \in \mathcal{I}$ and almost all t . We see that $w(t)$ does not depend on the disturbance $d(t)$. Since the output z satisfies $z(t) = J_2 w(t)$, we see that z does not depend on the disturbance either. Hence, system (9) is disturbance decoupled. \blacksquare

The subspace $\sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle$ plays an important role in our main results. Although each subspace $\langle A_j \mid \text{im } E_j \rangle$ is invariant under A_j , their sum $\sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle$ is not necessarily invariant under each A_j , so it is not always possible to use $\sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle$ as the subspace \mathcal{V} in Theorem 9. In the next lemma we give some conditions under which the subspace $\sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle$ is A_i -invariant for each $i \in \mathcal{I}$ and has a more compact form.

Lemma 10 *Let A_i and E_i satisfy (8). The subspace $\sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle$ is A_j -invariant and satisfies*

$$\sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle = \langle A_j \mid \mathcal{A} + \mathcal{E} \rangle \quad (19)$$

for each $j \in \mathcal{I}$ if one of the following conditions holds:

- i. $\mathcal{A} \subseteq \sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle$,
- ii. $\text{im}(M_j - M_i) \subseteq \sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle$ for all $i, j \in \mathcal{I}$,
- iii. $(M_j - M_i)(\text{im } F + C\mathcal{T}^*(A, E, C, F)) = \text{im}(M_j - M_i)$ for all $i, j \in \mathcal{I}$,
- iv. $F + C(sI - A)^{-1}E$ is right invertible.

Proof. We will prove this lemma by showing that $iv. \Rightarrow iii. \Rightarrow ii. \Rightarrow i. \Rightarrow (19)$. Define

$$\mathcal{V} := \sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle, \quad \mathcal{T}^* := \mathcal{T}^*(A, E, C, F).$$

($iv. \Rightarrow iii.$) If $F + C(sI - A)^{-1}E$ is right invertible, then using (6) we find that $\text{im } F + C\mathcal{T}^* = \mathbb{R}^{n_y}$, which implies the third condition.

($iii. \Rightarrow ii.$) From (5), we see that the subspace \mathcal{T}^* satisfies $\mathcal{T}^* = \mathcal{T}^*(A_i, E_i, C, F)$ for each $i \in \mathcal{I}$. Using (4) this gives us

$$\mathcal{T}^* \subseteq \langle A_i \mid \text{im } E_i \rangle,$$

for each $i \in \mathcal{I}$, which implies

$$A_i \mathcal{T}^* \subseteq A_i \langle A_i \mid \text{im } E_i \rangle \subseteq \langle A_i \mid \text{im } E_i \rangle \subseteq \mathcal{V}.$$

Furthermore, we have that

$$\text{im } E_i \subseteq \langle A_i \mid \text{im } E_i \rangle \subseteq \mathcal{V}$$

for all i in \mathcal{I} . This yields

$$\begin{aligned} (M_j - M_i)C\mathcal{T}^* &= (A_j - A_i)\mathcal{T}^* \subseteq \mathcal{V} \\ \text{im}(M_j - M_i)F &= \text{im}(E_j - E_i) \subseteq \mathcal{V} \end{aligned}$$

for any $i, j \in \mathcal{I}$. Together, this gives us

$$(M_j - M_i)(\text{im } F + C\mathcal{T}^*) \subseteq \mathcal{V}.$$

From the third condition it follows that $\text{im}(M_j - M_i) \subseteq \mathcal{V}$ for all $i, j \in \mathcal{I}$.

($ii. \Rightarrow i.$) : This follows from the fact that $\text{im}(A_j - A_i) = \text{im}(M_j - M_i)C \subseteq \text{im}(M_j - M_i)$.

($i. \Rightarrow (19)$) For any $i, j \in \mathcal{I}$ we have

$$\begin{aligned} A_j \langle A_i \mid \text{im } E_i \rangle &\subseteq A_i \langle A_i \mid \text{im } E_i \rangle \\ &\quad + (A_j - A_i) \langle A_i \mid \text{im } E_i \rangle \\ &\subseteq \langle A_i \mid \text{im } E_i \rangle + \text{im}(A_j - A_i) \\ &\subseteq \mathcal{V}, \end{aligned}$$

where we used $\mathcal{A} \subseteq \mathcal{V}$ in the last step. Hence, we see that

$$\begin{aligned} A_j \mathcal{V} &= A_j \left(\sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle \right) \\ &\subseteq \sum_{i \in \mathcal{I}} A_j \langle A_i \mid \text{im } E_i \rangle \subseteq \mathcal{V}, \end{aligned}$$

thus \mathcal{V} is A_j -invariant for every $j \in \mathcal{I}$. Since $\text{im } E_i \subseteq \mathcal{V}$ for all $i \in \mathcal{I}$ it follows that $\mathcal{E} \subseteq \mathcal{V}$, and hence $\mathcal{A} + \mathcal{E} \subseteq \mathcal{V}$. Consequently,

$$\langle A_j \mid \mathcal{A} + \mathcal{E} \rangle \subseteq \mathcal{V}, \quad (20)$$

since $\langle A_j \mid \mathcal{A} + \mathcal{E} \rangle$ is the smallest A_j -invariant subspace containing $\mathcal{A} + \mathcal{E}$.

For the other inclusion, note that

$$\begin{aligned} A_i \langle A_j \mid \mathcal{A} + \mathcal{E} \rangle &\subseteq A_j \langle A_j \mid \mathcal{A} + \mathcal{E} \rangle \\ &\quad + (A_i - A_j) \langle A_j \mid \mathcal{A} + \mathcal{E} \rangle \\ &\subseteq \langle A_j \mid \mathcal{A} + \mathcal{E} \rangle + \mathcal{A} \\ &\subseteq \langle A_j \mid \mathcal{A} + \mathcal{E} \rangle, \end{aligned}$$

which means that $\langle A_j \mid \mathcal{A} + \mathcal{E} \rangle$ is A_i -invariant for all $i \in \mathcal{I}$. Furthermore, we have $\text{im } E_i \subseteq \langle A_j \mid \mathcal{A} + \mathcal{E} \rangle$ for each $i \in \mathcal{I}$. Since $\langle A_i \mid \text{im } E_i \rangle$ is the smallest A_i -invariant subspace containing $\text{im } E_i$, we see that

$$\langle A_i \mid \text{im } E_i \rangle \subseteq \langle A_j \mid \mathcal{A} + \mathcal{E} \rangle,$$

for each $i \in \mathcal{I}$, and hence $\mathcal{V} \subseteq \langle A_j \mid \mathcal{A} + \mathcal{E} \rangle$. Together with (20) this completes the proof. \blacksquare

If one of the conditions in Lemma 10 is satisfied, we can combine Theorems 8 and 9 to obtain the following necessary and sufficient conditions for system (9) to be disturbance decoupled.

Corollary 11 *Assume that Assumptions 1, 2, and 7 are satisfied. If one of the conditions in Lemma 10 holds, then the linear multi-modal system (9) is disturbance decoupled if and only if*

$$\langle A_j \mid \mathcal{A} + \mathcal{E} \rangle \subseteq \ker J,$$

for every $j \in \mathcal{I}$.

5. Special classes of systems

In this section we revisit the examples discussed in Section 3 and apply Theorem 8, Theorem 9, and Corollary 11 to these systems. For the linear complementarity problem with $R = 0$, $H = 0$, and NG a symmetric positive definite matrix this will lead to new results, which are presented in Section 5.3.2. For the switched linear systems, conewise linear systems and the other linear complementarity problem, we compare our result with existing results in the literature.

5.1. Switched Linear Systems

The disturbance decoupling problem for switched linear systems has been studied in [39], in which a distinction is made between disturbance decoupling (DD) w.r.t. d and DD w.r.t. the switching signal σ . From Theorem 3.7 in [39] we see that system (10) is disturbance decoupled (w.r.t both d and σ) if and

only if there exists a subspace \mathcal{V} that is invariant under all A_i , satisfying

$$\text{im}(A_i - A_j) \subseteq \mathcal{V} \subseteq \ker J, \quad \text{im } E_i \subseteq \mathcal{V},$$

for all $i, j \in \mathcal{I}$.

The switched linear system satisfies Assumption 7 since every \mathcal{Y}_i equals \mathbb{R}^{n_y} . Therefore, we can apply Theorem 9, which gives the same sufficient condition as above for system (10) to be disturbance decoupled. However, the necessary condition we get from Theorem 8 is slightly weaker. This discrepancy can be explained by the observation that for switched linear systems the relative interior of every two cones \mathcal{Y}_i and \mathcal{Y}_j intersect, which means that for each $i, j \in \mathcal{I}$ there is a open neighborhood in \mathbb{R}^{n_y} in which the mode can change arbitrarily from i to j and back.

In the case that $\mathcal{A} \subseteq \sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle$, we have that $\sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle$ is the smallest subspace that contains $\text{im } E_i$ and is A_i -invariant for each $i \in \mathcal{I}$, and hence we could take $\sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E_i \rangle$ as the subspace \mathcal{V} in Theorem 3.7 in [39].

5.2. Conewise Linear Systems

Conewise linear systems can be seen as a special case of piecewise affine systems. In the case that $F = 0$, Corollary 3 in [14] shows that

$$\sum_{i \in \mathcal{I}} \langle A_i \mid \text{im } E \rangle \subseteq \ker J \quad (21)$$

is a necessary condition for the conewise linear system (12) to be disturbance decoupled. Corollary 5 in [14] states that system (12) is disturbance decoupled if there is a subspace $\mathcal{V} \subseteq \ker J$ that contains $\text{im } E$ and is invariant under each A_i . The necessary condition (21) can be recovered by Theorem 8, since Assumption 7 is satisfied, as each cone \mathcal{Y}_i is solid. From Theorem 9 we find that the existence of a subspace $\mathcal{V} \subseteq \ker J$ that is invariant under each A_i and contains $\text{im } E$ and \mathcal{A} is a sufficient condition for system (12) to be disturbance decoupled. This condition is stronger than the condition in Corollary 5 in [14]. This difference can be explained by the continuity assumption for the conewise linear system, which cannot be exploited for general linear multi-modal systems. In the case that $C(sI - A)^{-1}E$ is right invertible, then Corollary 9 in [14] yields (21) as a necessary and sufficient condition for disturbance decoupledness, which can be recovered by Corollary 11 in this paper.

A bimodal linear system is a special case of conewise linear systems. We consider the case that $y = c^T x$ for some vector c . It is shown in [15] that such a bimodal linear system is disturbance decoupled if and only if

$$\langle A_1 \mid \text{im } E \rangle + \langle A_2 \mid \text{im } E \rangle \subseteq \ker J,$$

even if $c^T(sI - A)^{-1}E$ is not right invertible. For this particular systems, the condition (i) of Lemma 10 holds regardless of whether $c^T(sI - A)^{-1}E$ is right-invertible or not. As such, the necessary and sufficient conditions for bimodal systems [15] can be recovered from Corollary 11.

5.3. Linear Complementarity Systems

5.3.1. Case 1

In Theorem 14.3 in [16] it is shown that linear complementarity system (14), with H being a P -matrix and the transfer matrix $R + N(sI - A)^{-1}E$ right invertible, is disturbance decoupled if and only if

$$\sum_{\alpha \in \mathcal{I}} \langle A_\alpha \mid \text{im } E_\alpha \rangle \subseteq \ker J. \quad (22)$$

This result can be recovered from Corollary 11. To see this, note that the cones \mathcal{Y}_i are solid for this linear complementarity system, which can be seen from the right-invertibility of $R + N(sI - A)^{-1}E$ and H being a P -matrix. Thus, Assumption 7 is satisfied, and since Assumptions 1 and 2 also hold (see Example 5), we can indeed apply Corollary 11 and find (22) as a necessary and sufficient condition for system (14) to be disturbance decoupled.

By exploiting the special relation between the matrices A_α and E_α , it is shown in Lemma 14.2 in [16] that

$$\sum_{\alpha \in \mathcal{I}} \langle A_\alpha \mid \text{im } E_\alpha \rangle = \langle A \mid \text{im } [E \quad G] \rangle.$$

Therefore, we find that

$$\langle A \mid \text{im } [E \quad G] \rangle \subseteq \ker J.$$

is a necessary and sufficient geometric condition for system (14) to be disturbance decoupled.

5.3.2. Case 2

We consider again the Linear Complementarity System (14), and now we assume that $R = 0$, $H = 0$, NG is a symmetric positive definite matrix and that the transfer matrix $N(sI - A)^{-1}E$ is right invertible as a rational matrix.

It turns out that checking that system (14) satisfies Assumption 7 requires more effort than case 1, as the cones \mathcal{Y}_i in this case are not all solid.

Lemma 12 *Suppose that $R = 0$, $H = 0$, NG is a symmetric positive definite matrix and that $N(sI - A)^{-1}E$ is right invertible. Then system (14) satisfies Assumption 7.*

Proof. We start with the observation that from the right-invertibility of the transfer matrix $N(sI - A)^{-1}E$ it follows that the transfer matrix

$$T_\alpha(s) := \begin{bmatrix} 0 \\ N_{\alpha \bullet} E \end{bmatrix} + \begin{bmatrix} N_{\alpha^c \bullet} \\ N_{\alpha \bullet} A \end{bmatrix} (sI - A)^{-1}E$$

is also right invertible for any $\alpha \subseteq \{1, 2, \dots, n_\eta\}$. Indeed, suppose that there is a rational vector $\text{col}(u_{\alpha^c}(s), u_\alpha(s))$ such that

$$\begin{bmatrix} u_{\alpha^c}^T(s) & u_\alpha^T(s) \end{bmatrix} T_\alpha(s) = 0.$$

Then, from the relation

$$s \cdot N_{\alpha \bullet} (sI - A)^{-1}E = N_{\alpha \bullet} E + N_{\alpha \bullet} A (sI - A)^{-1}E,$$

we see that

$$\begin{aligned} & \begin{bmatrix} u_{\alpha^c}^T(s) & \frac{1}{s} \cdot u_\alpha^T(s) \end{bmatrix} T_\alpha(s) \\ &= \begin{bmatrix} u_{\alpha^c}^T(s) & u_\alpha^T(s) \end{bmatrix} \begin{bmatrix} N_{\alpha^c \bullet} \\ N_{\alpha \bullet} A \end{bmatrix} (sI - A)^{-1}E = 0. \end{aligned}$$

The right-invertibility of $N(sI - A)^{-1}E$ implies that $\begin{bmatrix} u_{\alpha^c}^T(s) & u_\alpha^T(s) \end{bmatrix} = 0$. Consequently, $T_\alpha(s)$ is right invertible.

Next, we use (5) and (4) to observe that

$$\begin{aligned} & \mathcal{T}^*(A, E, \begin{bmatrix} N_{\alpha^c \bullet} \\ N_{\alpha \bullet} A \end{bmatrix}, \begin{bmatrix} 0 \\ N_{\alpha \bullet} E \end{bmatrix}) \\ &= \mathcal{T}^*(A_\alpha, E_\alpha, \begin{bmatrix} N_{\alpha^c \bullet} \\ N_{\alpha \bullet} A \end{bmatrix}, \begin{bmatrix} 0 \\ N_{\alpha \bullet} E \end{bmatrix}) \\ &\subseteq \langle A_\alpha \mid \text{im } E_\alpha \rangle. \end{aligned}$$

Since $T_\alpha(s)$ is right invertible we can use this together with (6) to find that

$$\text{im} \left[\begin{bmatrix} 0 \\ N_{\alpha \bullet} E \end{bmatrix} + \begin{bmatrix} N_{\alpha^c \bullet} \\ N_{\alpha \bullet} A \end{bmatrix} \right] \langle A_\alpha \mid \text{im } E_\alpha \rangle = \mathbb{R}^{n_\eta}$$

or, equivalently,

$$\begin{bmatrix} N_{\alpha^c \bullet} & 0 \\ N_{\alpha \bullet} A & N_{\alpha \bullet} E \end{bmatrix} (\langle A_\alpha \mid \text{im } E_\alpha \rangle \times \mathbb{R}^{n_d}) = \mathbb{R}^{n_\eta}. \quad (23)$$

We rewrite \mathcal{Y}_α as

$$\mathcal{Y}_\alpha = \begin{bmatrix} N & 0 \\ NA & NE \end{bmatrix} \tilde{\mathcal{Y}}_\alpha$$

with

$$\tilde{\mathcal{Y}}_\alpha = \left\{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} \geq 0 \right\} \cap \ker [N_{\alpha\bullet} \ 0],$$

where

$$\Theta_\alpha = \begin{bmatrix} I & 0 \\ 0 & -(N_{\alpha\bullet} G_{\bullet\alpha})^{-1} \end{bmatrix} \begin{bmatrix} N_{\alpha\bullet} & 0 \\ N_{\alpha\bullet} A & N_{\alpha\bullet} E \end{bmatrix}. \quad (24)$$

From (23) we see that

$$\Theta_\alpha (\langle A_\alpha \mid \text{im } E_\alpha \rangle \times \mathbb{R}^{n_d}) = \mathbb{R}^{n_\eta} \quad (25)$$

since the first matrix on the right-hand-side of (24) is non-singular. Hence Θ_α has full row rank, which gives us that

$$\text{rint} \left\{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} \geq 0 \right\} = \left\{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} > 0 \right\}.$$

Furthermore, (23) also shows that

$$\left\{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} > 0 \right\} \cap \ker [N_{\alpha\bullet} \ 0] \neq \emptyset. \quad (26)$$

Therefore, we can use Proposition 2.42 in [27] to find that the relative interior of $\tilde{\mathcal{Y}}_\alpha$ is given by

$$\text{rint } \tilde{\mathcal{Y}}_\alpha = \left\{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} > 0 \right\} \cap \ker [N_{\alpha\bullet} \ 0].$$

Note that

$$N_{\alpha\bullet} A_\alpha = 0, \quad N_{\alpha\bullet} E_\alpha = 0, \quad (27)$$

and hence $\langle A_\alpha \mid \text{im } E_\alpha \rangle \subseteq \ker N_{\alpha\bullet}$. Consequently, we have that

$$\begin{aligned} & (\langle A_\alpha \mid \text{im } E_\alpha \rangle \times \mathbb{R}^{n_d}) \cap \text{rint } \tilde{\mathcal{Y}}_\alpha \\ &= (\langle A_\alpha \mid \text{im } E_\alpha \rangle \times \mathbb{R}^{n_d}) \cap \left\{ \begin{bmatrix} x \\ d \end{bmatrix} \mid \Theta_\alpha \begin{bmatrix} x \\ d \end{bmatrix} > 0 \right\}, \end{aligned} \quad 355$$

which is non-empty, due to (25). Hence, the set

$$\begin{bmatrix} N & 0 \\ NA & NE \end{bmatrix} \left((\langle A_\alpha \mid \text{im } E_\alpha \rangle \times \mathbb{R}^{n_d}) \cap \text{rint } \tilde{\mathcal{Y}}_\alpha \right)$$

is also non-empty. Using Proposition 2.44(a) in [27], this implies that

$$\left(\begin{bmatrix} N \\ NA \end{bmatrix} \langle A_\alpha \mid \text{im } E_\alpha \rangle + \text{im} \begin{bmatrix} 0 \\ NE \end{bmatrix} \right) \cap \text{rint}(\mathcal{Y}_\alpha) \neq \emptyset,$$

which is Assumption 7(ii).

From (26) we know that there is a point \bar{y} such that $\Theta_\alpha \bar{y} > 0$ and $[N_{\alpha\bullet} \ 0] \bar{y} = 0$. This implies that for every $y \in [N_{\alpha\bullet} \ 0]$ there is a $\gamma \in \mathbb{R}$ such that $y + \gamma \bar{y} \in \tilde{\mathcal{Y}}_\alpha$, so $y \in \text{span}(\tilde{\mathcal{Y}}_\alpha)$. Hence, $\ker [N_{\alpha\bullet} \ 0] \subseteq \text{span}(\tilde{\mathcal{Y}}_\alpha)$. Together with $\tilde{\mathcal{Y}}_\alpha \subseteq \ker [N_{\alpha\bullet} \ 0]$ this gives us

$$\text{span}(\tilde{\mathcal{Y}}_\alpha) = \ker [N_{\alpha\bullet} \ 0].$$

With (27) this gives us

$$\langle A_\alpha \mid \text{im } E_\alpha \rangle \times \mathbb{R}^{n_d} \subseteq \text{span}(\tilde{\mathcal{Y}}_\alpha).$$

Hence,

$$\left(\begin{bmatrix} N \\ NA \end{bmatrix} \langle A_\alpha \mid \text{im } E_\alpha \rangle + \text{im} \begin{bmatrix} 0 \\ NE \end{bmatrix} \right) \subseteq \text{span}(\mathcal{Y}_\alpha),$$

and hence system (14) also satisfies Assumption 7(i). \blacksquare

Lemma 10 can not directly be applied to system (14), since the right-invertibility of $N(sI - A)^{-1}E$ does not imply that

$$\begin{bmatrix} 0 \\ NE \end{bmatrix} + \begin{bmatrix} N \\ NA \end{bmatrix} (sI - A)^{-1}E$$

is right invertible. However, the relation

$$N(sI - A)^{-1}E = \frac{1}{s} (NE + NA(sI - A)^{-1}E)$$

reveals that the right-invertibility of $N(sI - A)^{-1}E$ implies that $NE + NA(sI - A)^{-1}E$ is right invertible as well. So if we take $\tilde{C} = NA$ and $\tilde{F} = NE$ and write

$$A_\alpha = A + M_\alpha \tilde{C}, \quad E_\alpha = E + M_\alpha \tilde{F}$$

where

$$M_\alpha = -G_{\bullet\alpha} (N_{\alpha\bullet} G_{\bullet\alpha})^{-1} I_{\alpha\bullet},$$

then condition (iv) of Lemma 10 holds with C and F replaced by \tilde{C} and \tilde{F} respectively. Consequently, $\sum_{\alpha \in \mathcal{I}} \langle A_\alpha \mid \text{im } E_\alpha \rangle$ is A_α -invariant for all α , and contains \mathcal{A} . Therefore, we can apply Corollary 11 to system (14). Before we do so, we first find a more compact form of $\sum_{\alpha \in \mathcal{I}} \langle A_\alpha \mid \text{im } E_\alpha \rangle$.

Lemma 13 *Suppose that $R = 0$, $H = 0$, NG is a symmetric positive definite matrix and that $N(sI - A)^{-1}E$ is right invertible. Then we have*

$$\sum_{\alpha \in \mathcal{I}} \langle A_\alpha \mid \text{im } E_\alpha \rangle = \langle A \mid \text{im} [E \ G] \rangle.$$

Proof. From the discussion above and taking $\alpha = \emptyset$ in Lemma 10, we see that

$$\sum_{\alpha \in \mathcal{I}} \langle A_\alpha \mid \text{im } E_\alpha \rangle = \langle A \mid \mathcal{A} + \mathcal{E} \rangle, \quad (28)$$

with \mathcal{A} and \mathcal{E} as in (18). Note that $\text{im } E_\alpha \subseteq \text{im } [E \ G]$ for each $\alpha \in \mathcal{I}$, so $\mathcal{E} \subseteq \text{im } [E \ G]$. Furthermore, for every $\alpha, \beta \in \mathcal{I}$ we have $\text{im}(A_\beta - A_\alpha) \subseteq \text{im } G$, and hence $\mathcal{A} \subseteq \text{im } G$. So we can conclude that $\mathcal{A} + \mathcal{E} \subseteq \text{im } [E \ G]$.

To prove that the other inclusion also holds, choose $\alpha = \{1, 2, \dots, n_\eta\}$ and $\beta = \emptyset$, then we see that

$$[A_\beta - A_\alpha \quad E_\beta - E_\alpha] = G(NG)^{-1} [NA \quad NE].$$

The right-invertibility of $NE + NA(sI - A)^{-1}E$ implies that $[NA \quad NE]$ is of full row rank, and since NG is symmetric positive definite, this implies that

$$\text{im } [A_\beta - A_\alpha \quad E_\beta - E_\alpha] = \text{im } G,$$

and hence $\text{im } G \subseteq \mathcal{A} + \mathcal{E}$. By taking $\alpha = \emptyset$ we find that $\text{im } E \subseteq \mathcal{E}$, and hence $\text{im } [E \ G] \subseteq \mathcal{A} + \mathcal{E}$.

Together, this gives us

$$\mathcal{A} + \mathcal{E} = \text{im } [E \ G],$$

which, combined with (28), proves the statement. ■

Now, combining Corollary 11 with Lemma 12 and Lemma 13, we have the following result for the linear complementarity system (14).

Theorem 14 *Suppose that $R = 0$, $H = 0$, NG is a symmetric positive definite matrix and that $N(sI - A)^{-1}E$ is right invertible as a rational matrix. Then the linear complementarity system (14) is disturbance decoupled if and only if*

$$\langle A \mid \text{im } [E \ G] \rangle \subseteq \ker J.$$

Here we see that, although system (14) is highly non-linear and non-smooth, the conditions for system (14) to be disturbance decoupled are geometric in nature and very akin to those for linear systems, for which $\langle A \mid \text{im } E \rangle \subseteq \ker J$ is the condition. For the linear complementarity system we see that the effect of the complementarity variables on the state, captured by $\langle A \mid \text{im } G \rangle$, also has to be taken into account.

6. Conclusion and discussion

In this paper, we presented necessary and sufficient conditions, geometric in nature, under which a general linear multi-modal system is disturbance decoupled. The main results, presented in Theorem 8, Theorem 9, and Corollary 11 generalizes all existing results in the literature on switched linear systems [39], bimodal systems [15], conewise linear systems [14], linear complementarity systems with P -property [16]. In addition, these results lead to necessary and sufficient conditions for a class of linear complementarity systems (see Theorem 14) whose disturbance decoupling properties were not been studied before.

For the presented general linear multi-modal system the necessary condition in Theorem 8 and the sufficient condition in Theorem 9 for being disturbance decoupled do not coincide. In Corollary 11 we presented several conditions under which these conditions do coincide.

In this paper we only studied under what conditions a general linear multi-modal system is disturbance decoupled; rendering a system disturbance decoupled by means of feedback is the next step. Finding a static state feedback such that the resulting closed-loop system satisfies (15) becomes a linear algebraic problem and can be solved mimicking the footsteps for the linear case.

Possible future research lines include extending the results presented in this paper to piecewise affine systems and to study the extension to Filippov solutions. Furthermore, the results for the linear complementarity systems might be extended to the more general case with a not necessarily symmetric but positive semi-definite H for which there exists a positive symmetric matrix K such that $KG u = N^T u$ for all $u \in \ker(H + H^T)$.

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